General Theory of Relativity

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1 Motivation: Equivalence Principle

Newtons law of motion in the gravitational field:

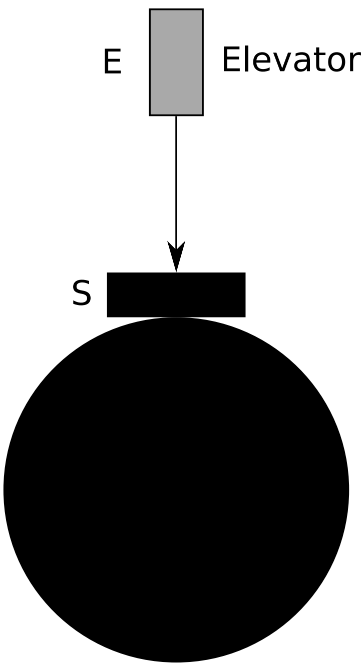
$$m_I \cdot \ddot{\vec{x}} = m_G \cdot \vec{g} \tag{1}$$

where m_I is the intertial mass and m_G is the gravitational mass.
Consider for analogy the coulomb law:

$$m_I \cdot \ddot{\vec{x}} = q \cdot \vec{F_{el}} \tag{2}$$

In this case q takes the role of m_G, but it's definitely $q \neq m_I$ in contrast to $m_I = m_G$.

1.1 Elevator: Gedankenexperiment



Consider an elevator falling down towards the surface of the earth. The obsever in the elevator be in the intertial frame **E**, the observer on the surface of the earth in the inertial frame **S**:

- m_I^E: Inertial mass of the observer on the earth
- m_G^E Gravitational mass of the observer on the earth
- m_I^S: Inertial mass of the observer in the elevator
- m_G^S Gravitational mass of the observer in the elevator
- $\vec{g}$: Gravitational field of the earth

With Newtons law then is:

$$m_I^E \cdot \ddot{\vec{x}}_E = m_G^E \cdot \vec{g} \tag{3}$$

Experimental evidence: An observer freely falling in the gravitational field of the earth doest not feel any inertial forces.

$\Rightarrow$ The elevator observer imagines himself in an inertial frame.

From the point of **E** the earth's surface is accelerating with $\ddot{\vec{x}}_S = -\ddot{\vec{x}}_E$ towards the observer in **E**. **E** therefore predicts that the observer in **S** feels an ineratial force

$$\vec{F}_I^S = -m_I^S \cdot \ddot{\vec{x}}_s = m_I^S \ddot{\vec{x}}_E = m_I^S \frac{m_G^E}{m_I^E} \vec{g} \tag{4}$$

This is interpreted by **S** as the gravitational force $\vec{F}^S_G = m^S_G \cdot \vec{g}$ that presses **S** onto the surface:

$$\vec{F}^S_G = \vec{F}^S_I \Leftrightarrow \frac{m^S_G}{m^S_I} = \frac{m^E_G}{m^E_I} =: \mu \tag{5}$$

We can reabsorb μ into $\vec{g}$ which then leads to

$$m_G = m_I \tag{6}$$

Consequence of the equivalence principle:

- „Gravitational fields and inertial forces are one and the same thing“
- „A freely falling observer is (locally) in an inertial frame“
  (The gravitational field is in general inhomogeniuos, so if the elevator is ls too large on does see tidal foces)

To get the actual equation of motion in a gravitational field we recall some elements of **Special Relativity**.
Statements about observations of inertial observers (I think observers in inertial frames) moving relatively to each other at constant speed v in x-direction:

- $\mathbf{I}$: t, x coordinates inertial frame
- $\mathbf{I}'$: t', x' coordinates inertial frame

If $\mathbf{I}'$ moves at speed v along the x-axis then the origin of $\mathbf{I}'$ as measured by $\mathbf{I}$ has moved by a distance $v \cdot t$ after time lapse t.
Simultaneity: Since the speed of light is finite it is practically impossible to synchronise all clocks, simultanious events now must be carefully defined.
Experimental evidence: Speed of light c does not depend on the speed of motion v.
Definition: Two events $(t_1, x_1), (t_2, x_2)$ are called simultanious if the ligth signals emitted from them meet in the middle:

$$\frac{\overline{AC}}{\overline{AB}} = \frac{\overline{AE}}{\overline{AD}} = 2 \tag{7}$$

$$\Rightarrow \overline{AF} = \overline{CF} \tag{8}$$

because triangle ACF is equal sided by one of the congruence laws.

$$\Rightarrow \alpha = \beta \ , \ \tan(\beta) = \frac{v}{c} \tag{9}$$

To take into account that beam of $x' = 0$ must be $\sim x - v \cdot t$ we must have $x' = a \cdot (x - v \cdot t)$.
To take into account that beam of $t' = 0$ with $\sim t - c \cdot t \frac{v}{c} x$ we must have $t' = b \cdot \left(t - \frac{v}{c^2} x\right)$ match the linear beam $\mathbf{I} \leftrightarrow \mathbf{I}'$ with the result of $v \leftrightarrow -v$.
To determine a and b:

- A light ray $x = c \cdot t$ in $\mathbf{I}$ and $x' = c \cdot t'$ in $\mathbf{I}'$:

$$\frac{x'}{t'} = \left. \frac{a \cdot (x - v \cdot t)}{b \cdot (t - \frac{v}{c^2})} \right|_{x = c \cdot t} = \frac{a}{b} c = c \Rightarrow a = b \tag{10}$$

- Invert transformation:

$$x' = a \cdot (x - v \cdot t) \Rightarrow x = \frac{1 - \left(\frac{v}{c}\right)^2}{a} \cdot (x' + v \cdot t') \stackrel{!}{=} a \cdot (x' + v \cdot t') \tag{11}$$

$$t' = a \cdot \left(t - \frac{v}{c^2} x\right) \Rightarrow t = \frac{1 - \left(\frac{v}{c}\right)^2}{a} \cdot \left(t' + \frac{v}{c^2}\right) \stackrel{!}{=} a \cdot \left(t' + \frac{v}{c^2} x'\right) \tag{12}$$

$$\Rightarrow a = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \tag{13}$$

With that said we write the transformation as followed:

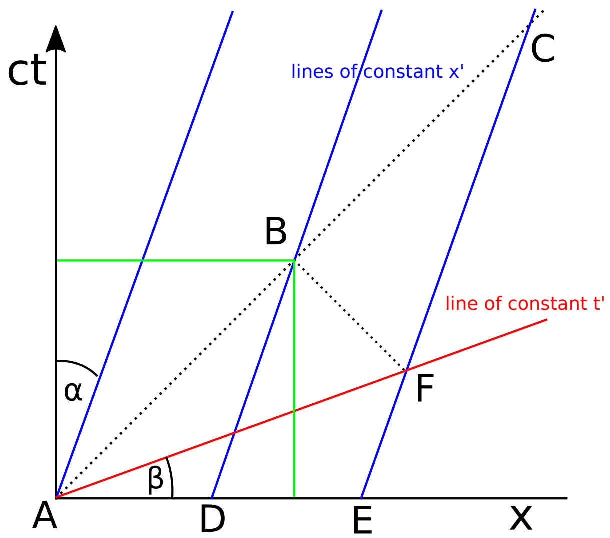
$$x' = \gamma \cdot (x - v \cdot t) \tag{14}$$

$$t' = \gamma \cdot \left(t - \frac{v}{c^2} x\right) \tag{15}$$

$$y' = y \tag{16}$$

$$z' = z \tag{17}$$

This is called the **Lorentz-Transformation** and gives raise to various effects like time dilatation, length contraction and much more.
In what follows we are interested in the time dilatation effect.



Time dilatation effect: We consider a clock which rests in $\mathbf{I'}$ and which passes in $\mathbf{I}$ the points $x_A = 0$ and $x_B = v \cdot (t_B - t_A)$ after the time by $\mathbf{S}$ $t_B - t_A$ as measured by a clock in $\mathbf{I}$.
Two events A and B:

$$x'_A - x'_B = \gamma \cdot ((x_A - x_B) - v \cdot (t_A - t_B)) = 0 \tag{18}$$

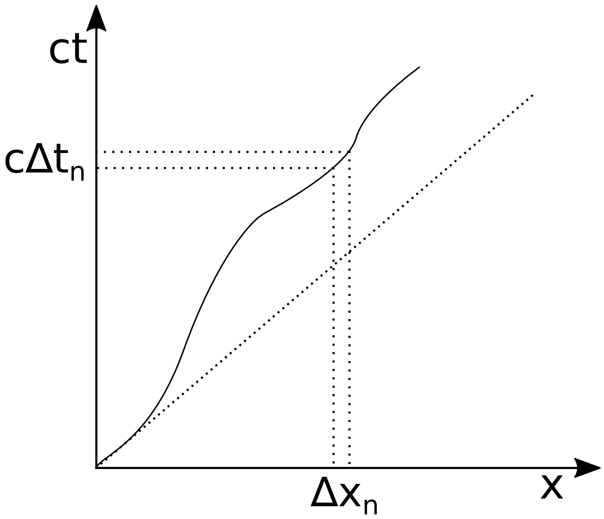
Clock rests in $\mathbf{I'}$: (19)

$$(t'_B - t'_A) = \gamma \cdot ((t_B - t_A) - \frac{v}{c^2}(x_B - x_A)) = \tag{20}$$

$$\gamma \cdot \left(1 - \left(\frac{v}{c}\right)^2\right) \cdot (t_B - t_A) = \frac{1}{\gamma} \cdot (t_A - t_B) \tag{21}$$

thus many clocks run at a slower speed.

1.2 Line Element



We consider a general trajectory. We want to know how much time passes for the observer that considers herself at rest when moving along the trajectory.

$$\Delta t'_n = \frac{1}{\gamma_n}\Delta t_n \tag{22}$$

$$\text{where } \gamma_n = \frac{1}{\sqrt{1-\left(\frac{v_n}{c}\right)^2}}, \tag{23}$$

$$v_n = \dot{x}(t_n) \tag{24}$$

and $t \mapsto x(t)$ is trajectory as seen by **I**. If we consider the time lapse T in **I** then for the moving observer one measures

$$T^1 = \lim_{n\to\infty}\sum_{n=1}^{n}\Delta t'_n \ , \ \Delta t = \frac{T}{N} = \int_0^T \mathrm{d}t\sqrt{1-\frac{(\dot{x}(t))^2}{c^2}} \tag{25}$$

which is called **Eigenzeit** of the observer. Here we have done this infinite many times for motion in x-direction, for a more general curve in 3d it's easy to see that T^1 generalises to

$$T^1 = \int_0^T \mathrm{d}t\sqrt{1-\frac{\left(\dot{\vec{x}}(t)\right)^2}{c}} = \frac{1}{c}\int_0^T |\,\mathrm{d}s| \tag{26}$$

with the line element

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \delta_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b \quad \forall a,b\in\{1,2,3\} \tag{27}$$

$$\Rightarrow cT^1 = \int \sqrt{\mathrm{d}x^\mu\,\mathrm{d}x^\nu\eta_{\mu\nu}} \ \text{ with } \eta_{00}=-1,\ \eta_{ab}=\delta_{ab},\ \eta_{0a}=0,\ x^0 = c\cdot t \tag{28}$$

reminds of the formula for the Euler length of a curve:

$$L = \int \sqrt{\delta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} \tag{29}$$

⇒ Eigentime can be interpreted as the „length" of a spacetime curve but not with respect to the Euclidean metric which we use to measure spatial distances but rather with respect to the Minkowski metric.

The twin paradox is not a paradox beause the situation is not symmetric under exchange of ineratial observer (t,x) and rocket observer (t',x') who is not in an inertial frame, therefore the formula

$$T = \int_0^{T^1} \mathrm{d}t'\sqrt{1-\frac{v'(t')^2}{c^2}} \tag{30}$$

A What however is certainly possible is to do the following: We find four maps $\varphi^\mu : \mathbb{R} \to \mathbb{R}^4$

$$(y^\mu)^3_{y=3} \mapsto \varphi^\mu(y)\varphi^\mu(y) = x^\mu \tag{31}$$

such that the straight line trajectory $y^1 = const, y^2 = const, y^3 = const, y^0 = c \cdot t$ let's call it $y_\mu(t)$ is mapped under φ into the given trajectory $x^\mu(t) = \varphi^\mu(y(t))$.
The trajectory $y(t)$ would be the trajectory that the observer in the rocket would associate with herself.

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \left[\eta_{\mu\nu}\frac{\partial\varphi^\mu(y)}{\partial y^\varrho}\frac{\partial\varphi^\nu(y)}{\partial y^\sigma}\right]\mathrm{d}y^\varrho\,\mathrm{d}y^\sigma \tag{32}$$

$$T^1 = \int_0^T \mathrm{d}t\sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} = \int_0^T \mathrm{d}t\sqrt{-g_{\mu\nu}(y)\,\mathrm{d}y^\mu\,\mathrm{d}y^\nu} \tag{33}$$

T^1 does not depend on which one we use but we see 2 interesting effects.

- In general coordinates we must use a coordinate dependent metric, i.e. a metric field $y \mapsto g(y)$ rather than a constant tensor g
- We have seen by the equivalence principle that accelerated motion is equivalent to a gravitational field
$\Rightarrow$ It is motivated to asssume that gravitational effects can be encoded in terms of a metric field. This metric field will be of a more general type than the one discovered in the twin paradox:

$$g_{\mu\nu}(y) = \eta_{\varrho\sigma}\frac{\partial\varphi^\varrho(y)}{\partial y^\mu}\frac{\partial\varphi^\sigma(y)}{\partial y^\nu} = (\varphi^*_\eta)_{\mu\nu}(y) \tag{34}$$

This is a 'fake' gravitational field because by choosing appropriate coordinates (here the one's of the earth observer) we can go back globally in spacetime to Minkoswki metric.
Pull-back of η by the diffeomorphism:

$$\varphi \;:\; \mathbb{R} \mapsto \mathbb{R} \tag{35}$$

Conversely, consider arbitrary metric field

$$g_{\mu\nu}(y) = g_{\mu\nu}, \det(g) \neq 0 \tag{36}$$

can always be transformed into the form

$$\eta_{\mu\nu} = S^\varrho_\mu(y) g_{\varrho\sigma}(y) S^\sigma_\nu(y) \tag{37}$$

by choosing an appropriate matrix field S, but S does not have to come from diffeomorphism.
The motion of a particle with rest mass m and charge q is encoded by the action:

$$S = \frac{q}{c}\int_{r_0}^{r_1} \mathrm{d}r A_\mu(x(r))\frac{\mathrm{d}x^\mu(r)}{\mathrm{d}r} - mc\int_{r_0}^{r_1}\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu(r)}{\mathrm{d}r}\frac{\mathrm{d}x^\nu(r)}{\mathrm{d}r}} \tag{38}$$

Extremising S results in the e.o.m. (equation of motion)

$$\frac{q}{c}\left(\frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu\right)(x(t))\frac{\mathrm{d}x^\nu}{\mathrm{d}r}(r) = mc\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\frac{\mathrm{d}x_\mu(r)}{\mathrm{d}r}}{\sqrt{-\frac{\mathrm{d}x_\mu(r)}{\mathrm{d}r}\frac{\mathrm{d}x^\mu(r)}{\mathrm{d}r}}}\right) \tag{39}$$

$$\Leftrightarrow m\ddot{\vec{x}} = q(\vec{E} + \vec{x}\times\vec{B}), \quad \text{for } \frac{||\dot{\vec{x}}||}{c} << 1,\ E^a = F^{0a},\ B^a = \frac{1}{2}\epsilon^{abc}F_{bc} \tag{40}$$

Motion of test particles in external el. mag. fields can be derived from alone actions. So following our assumption we postualte that the motion of test particles in external gravitational field $g_{\mu\nu}$ can be described by

$$S = -mc\int_{r_0}^{r_1}\mathrm{d}r\underbrace{\sqrt{-g_{\mu\nu}(x(t))\frac{\mathrm{d}x^\mu(r)}{\mathrm{d}r}\frac{\mathrm{d}x^\nu(r)}{\mathrm{d}r}}}_{L} \tag{41}$$

Extremising the action (Euler-Langrange Equ.)

$$\frac{d}{dr}\left(\frac{\partial L}{\partial\dot{x}^\mu}\right) = -mc\left(\frac{-g_{\mu\nu}\dot{x}^\nu}{w}\right) = \frac{\partial L}{\partial x^\mu} = -mc\left[-\frac{1}{2}\frac{g_{\varrho\sigma,\mu}\dot{x}^\varrho\dot{x}^\sigma}{w}\right] \tag{42}$$

$$\frac{d}{dr}\left(\frac{g_{\mu\nu}\dot{x}^\nu}{w}\right) = \frac{1}{2}\frac{g_{\varrho\sigma,\mu}\dot{x}^\varrho\dot{x}^\sigma}{w} = \frac{g_{\mu\nu,\varrho}\dot{x}^\varrho\dot{x}^\nu}{w} + \frac{g_{\mu\nu}\ddot{x}^\nu}{w} + g_{\mu\nu}\dot{x}^\nu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{w}\right) \tag{43}$$

$$g_{\mu\nu}\ddot{x}^\nu + \frac{1}{2}\left[g_{\mu\nu,\varrho} + g_{\mu\varrho,\nu} - g_{\nu\varrho,\mu}\right]\dot{x}^\nu\dot{x}^\varrho = \frac{\dot{w}}{w}g_{\mu\nu}\dot{x}^\nu \tag{44}$$

Geodesic equation of motion of a test particle in an external gravitational field in a general parametrisation $r \mapsto x^\mu(r)$ of the trajectory. The choice of parametrisation is arbitrary because S is reparametrisation invariant: If we write $r = f(r')$ where f is strictly monotonous and $x'(r') = x(f(r'))$ then $S[x'] = S[x]$. There are distinguished parametrisations for which the **rhs** (right hand side) vanishes, called affine parametrisation.

1.3 Newtonian Limit

Assumptions:

- $\frac{||\dot{\vec{x}}||}{c} << 1$
- $g_{\mu\nu} - \eta_{\mu\nu} =: 2h_{\mu\nu}$

Newton does not take special relativity into account.
Virial theorem:
$h_{\mu\nu}$ will encode the potential energy and thus in average will be of the same order as

the kinetic one
Approximating the geodesic e.o.m. up to linear order in $\frac{||\dot{\vec{x}}||}{c}, h_{\mu\nu}$

$$\ddot{x}^\mu + \Gamma^\mu_{\varrho\sigma}\dot{x}^\varrho\dot{x}^\sigma = \frac{\dot{w}}{w}\dot{x}^\mu, \quad g^{\mu\nu}\Gamma_{\nu\varrho\sigma} =: \Gamma^\mu_{\varrho\sigma} \tag{45}$$

$$\frac{w^2}{c^2} = -g_{\mu\nu}\frac{\dot{x}^\mu\dot{x}^\nu}{c^2} = \frac{1}{c^2}\left(c^2 - \dot{\vec{x}}^2 - 2h_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) = \left(1 - \left(\frac{||\dot{\vec{x}}||}{c}\right)^2 - 2h_{\mu\nu}\frac{\dot{x}^\mu\dot{x}^\nu}{c^2}\right) \tag{46}$$

$$\ddot{x}^a + \Gamma^a_{\varrho\sigma}\dot{x}^\varrho\dot{x}^\sigma = \frac{1}{2}\frac{\dot{w}}{w}\dot{x}^a \tag{47}$$

Inverse metric tensor field:

$$g^{\mu\nu} \cdot g_{\nu\sigma} = \delta^\mu_\sigma \tag{48}$$

$$g^{\mu\nu} = \eta^{\mu\nu} = 2h^{\mu\nu} \tag{49}$$

$\Gamma^a_{\varrho\sigma}\dot{x}^\varrho\dot{x}^\sigma$ already of first order

$$\Rightarrow \Gamma^a_{\varrho\sigma}\dot{x}^\varrho\dot{x}^\sigma = \Gamma^a_{00}c^2 + 0(hx^2) = \Gamma_{a00}c^2 + 0(hx^2) \tag{50}$$

$$2\Gamma_{a00} = h_{a0,0} + h_{a0,0} - h_{00,a} = \frac{2\dot{h}_{a0}}{c} - h_{00,a} \tag{51}$$

In Newtonain physics: An external gravitational field generated by a mass M gives rise to a potential $U = \frac{GM}{r}$.

$$\ddot{x}^a = -\nabla_a U \Rightarrow h_{00}c^2 = -U = \frac{GM}{r} \tag{52}$$

$$h_{00} = \frac{GM}{c^2 r} \tag{53}$$

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + \mathrm{d}\vec{x}^2 + 2h_{00}(\mathrm{d}x^0)^2 = -\left(1 - \frac{2GM}{c^2}\frac{1}{c}\right)(\mathrm{d}x^0)^2 + \mathrm{d}\vec{x}^2 \tag{54}$$

Where we call

$$R = \frac{2GM}{c^2} \tag{55}$$

the Schwartzschild radius.
Is this an allowed metric:
The metric has to follow certain equations called the Einstein field equations which are the direct analogon to the Maxwell equations for the electromagentic field. These equations say that $\mathrm{d}s^2$ has to be connected:

$$\mathrm{d}s^2 = -\left(1 - \frac{R}{r}\right)(\mathrm{d}x^0)^2 + \frac{\mathrm{d}r^2}{1 - \frac{R}{r}} + r^2(\mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\varphi^2) \tag{56}$$

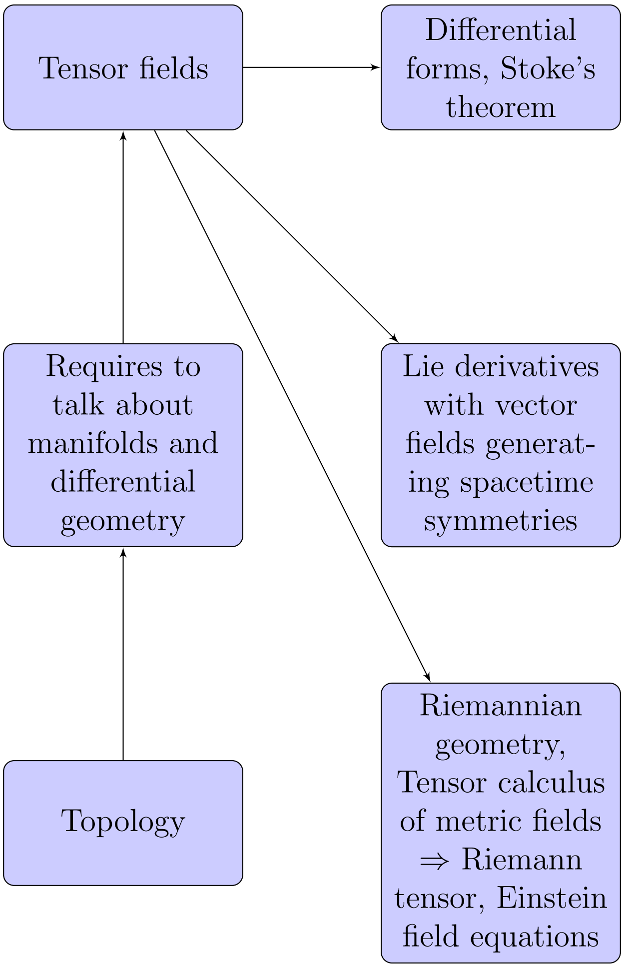
$$\mathrm{d}s^2_{Minkowski} = -(\mathrm{d}x^0)^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin(\theta)^2\,\mathrm{d}\varphi^2) \tag{57}$$

Einstein field equations can be derived from an action principle again similar to the electromagnetic case where the corresponding Lagrangian given by

$$L \sim F_{\mu\nu} F_{\varrho\sigma} \eta^{\mu\varrho} \eta^{\nu\sigma} \ , \ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{58}$$

depends on first derivatives of A_μ. If we would make an educated guess, we would probably say that the gravitational Lagrangian will be a quadratic expression of the first derivatives of the gravitational potential $g_{\mu\nu}$, i.e. $g_{\mu\nu,\varrho} \sim \Gamma_{\mu\nu\varrho}$ i.e. a Lagrangian $\sim \Gamma^2$. This guess is almost correct. the actual Lagrangian depends on the so called **Ricei Scalar** which depends next to Γ^2 also on $\partial\Gamma$.

1.4 Plan for the next few weeks



2 Mathematical properties

2.1 Elements of topology

Motivation: Differential manifolds are topological spaces of a particular type

- topology: about convergence, closeness, continuity of functions in an abstract setting in abstact spaces in which there is no distanc
- in particular advanced
- relevant in quantum mechanichs (topology of Hilbert spaces)

Definition
Let X be a set. A collection $\mathcal{U}$ of subsets of X is called a topology on X if

1. $\varnothing$, $X \in \mathcal{U}$
2. $\mathcal{U}$ is closed under finite intersections

$$U_1, \ldots, U_n \in \mathcal{U} \Rightarrow \bigcap_{k=1}^{n} U_k \in \mathcal{U} \tag{59}$$

3. U is closed under arbitrary unions (even uncountable)

$$U_\alpha \in \mathcal{U}, \alpha \in I \text{ arbitrary index sets } \Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{U} \tag{60}$$

The sets $U \in \mathcal{U}$ are called open, their complements $X - U$ are called closed.

Definition

- $N \subset X$ is called neighbourhood of $x \in X$
$\Leftrightarrow \exists U \in \mathcal{U} \ni x \in U \subset N$
- A base $\mathcal{B}$ of $\mathcal{U}$ is such that any $U \in \mathcal{U}$ is an arbitrary union of elements from $\mathcal{B}$
- A neighbour base $\mathcal{N}$ at $x \in X$ is a family of neighbourhoods of x such that for any neighbourhood M of x:

$$\exists N \in \mathcal{N} \ni N \subset M \tag{61}$$

- A topology $\mathcal{U}$ on X is called stronger (finer) than a topology $\mathcal{U}'$ on X if $\mathcal{U}' \subset \mathcal{U}$. $\mathcal{U}'$ is then called weaker (coarser) than $\mathcal{U}$.

Definiton Suppose that $(X,\mathcal{U})$ is a topological space and $Y \subset X$ is a subset. Then Y carries a natural topology

$$\mathcal{U}_Y = \{U \cap Y,\ U \in \mathcal{U}\} \tag{62}$$

called the induced (relative) topology on Y inherited from X. (Notation: $Y \hookrightarrow X$)
Definition

1. A function $f : X \to Y$ between two topological spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is called continuous

$$\Leftrightarrow f^{-1}(\mathcal{V}) \subset \mathcal{U} \tag{63}$$

$$\tag{64}$$

i.e. $f^{-1}(V) \in \mathcal{U}\ \forall\ V \in \mathcal{V}$
(here $f^{-1}(V) = \{x \in X; f(x) \in V\}$ is the pre-image of V, f^{-1} has nothing to do with the inverse of f which may not even exist.

2. However, if f is a bijection and f^{-1} is also continuous then f is called a homeomorphism (topological isomorphism).

Lemma:

$$f \text{ is continuous } \Leftrightarrow f \text{ is continuous at every point } x \in X \tag{65}$$

$$\Leftrightarrow \forall V \text{ open nbh of } f(x) \exists \text{ open nbh } U \text{ of } x \ni V \forall x' \in U \tag{66}$$

$$\text{i.e. } f(U) \subset V$$

Remark A topology is therefore defined by specifying which sets are open or equivalently which functions are continuous. In metric space one can also define a topology by saying which sequences are convergent, but this no longer time in general topological space and one must generalise to the notion of "nets".

To get interesting spaces one usually adds separation, coutability and compactness properties.
Definiton

- A topological space X is called disconnected if it is disjoiunt union of at least 2 non-empty subsets
- A topological space is called **Hausdorff**

$$\Leftrightarrow \forall x \neq y; x, y \in X \exists \text{ open nbh } U, V \text{ of } x, y \ni U \cap V = \varnothing \tag{67}$$

- A topological space X is called separable $\Leftrightarrow$ $\exists$ countable dense subset $S \subset X$ (Every neighbourhood N of any point $x \in X$ contains at least one point in S.

- A topological space X is called first countable if every point $x \in X$ has a countable nbh. base and second countable if it has a inverse countable base.

- A topological space X is called compact $\Leftrightarrow$ any open cover ξ of X has a finite subcover (i.e. $\exists c1, \dots c_n \ni \cup_{k=1}^{n} c_k = X$)

Remarks

- Hausdorff property is a separation property called T_2 in some books, $\exists$ different such notions denoted T_1, T_2, T_3, T_4 in books of topology.

- Suppose X is a metric space

$$\mathrm{d} : X \times X \to \mathbb{R}^+ \tag{68}$$
$$\to \mathrm{d}(x, y) = 0 \Leftrightarrow x = y \tag{69}$$
$$\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z) \forall x, y, z \tag{70}$$
$$\mathrm{d}(x, y) = \mathrm{d}(y, x) \tag{71}$$

  Open balls

$$\mathcal{O}_{\varepsilon}(x) = \{y \in ; \mathrm{d}(x, y) < \varepsilon\} \tag{72}$$

  generate topology on X by forming all unions and finite intersections of these; get topology on X defined by d.

Lemma: Metric spaces are always first countable and second countable only if separable (second countable $\Rightarrow$ first countable).

To heave meaningful notions of convergence in general topological spaces we need the notopn of “nets”.

Definition

- A partial order „$\geq$“ on a set A is a relation on A (i.e. a subset of $A \times A$ which is reflexive ($\alpha \leq \alpha$), symmetric ($\alpha \leq \beta \wedge \beta \leq \alpha \Rightarrow \alpha = \beta$) and transitive ($\alpha \leq \beta \wedge \beta \leq \gamma \Rightarrow \alpha \leq \gamma$))

  Note: not all $\alpha, \beta \in A$ nedd to be in relation but if they are then A is acalled linearly ordered by $\leq$

- A partially ordered set A is called directed $\Leftrightarrow \forall \alpha, \beta \exists \gamma \ni \alpha \leq \gamma, \beta \leq \gamma$

- A net in X is a map $A \to X, \alpha \mapsto x_{\alpha}$ from a partially ordered + directed set A into X

- A net $(x_{\alpha})_{\alpha \in A}$ converges to $x \in X$, denoted $\lim_{\alpha} x_{\alpha} = x \Leftrightarrow \forall U$ open nbh. U of $x \exists \alpha(U) \in A \ni x_{\alpha} \in U \forall \alpha \geq \alpha(U)$

- A subnet $(x_{\alpha(\beta)})_{\beta\in B}$ is defined by a map $\alpha : B \to A, \beta \to \alpha(\beta)$ between partially ordered and directed sets A, B

$$\ni \forall\alpha_0 \in A \exists\beta(\alpha_0) \in B \tag{73}$$
$$\ni \alpha_0 \leq \alpha\beta\forall\beta(\alpha_0) \leq \beta \tag{74}$$

Remarks

- Nets generalis the notion of sequences whic hare special nets
- Many theorems familiar from analysis do not generalise to arbitrary topological spaces if one sticks to sequences. However they do when one uses the notion of nets

Theorem: A convergent function $f : X \to Y$ is continuous
$\Leftrightarrow \forall$ net $(x_\alpha)_{\alpha\in A}$ in X the net $(y_\alpha)_{\alpha\in A}$ in Y with $y_\alpha = f(x_\alpha)$ is also convergent.
In particular if $\lim_\alpha x_\alpha = x$ then $\lim_\alpha f(x_\alpha) = f(x)$. ($f$ continuous $\Rightarrow f$ sequence convergent but "$\Leftarrow$" does not hold in general)

One can show that sequence convergence is sufficient if x, y are first countable spaces. A sequence $(x_n)_{n\in\mathbb{N}}$ may have a cluster point x but $\nexists$ subsequence converging to it. (x is the limit of a subnet of $(x_n)_{n\in\mathbb{N}}$)
Also the following theorem goes wrong in general topological spaces when sticking to sequences:
Theorem: (Bolzano-Weierstraß)
A topological space is compact $\Leftrightarrow$ every net $(x_\alpha)_{\alpha\in A}$ in X has a convergent subnet $\left(x_{\alpha(\beta)}\right)_{\beta\in B}$
Theorem
$Y \subset X$ is a closed subset $\Leftrightarrow$ if $(x_\alpha)_{\alpha\in A}$ converges in X to $x \in X$ and $X_\alpha \in Y \forall\alpha \in A$ then actually $x \in Y$.
Theorem

- Closed subset of compact spaces are compact
- Continuous images of compact sets are compact
- Compact subsets of Hausdorff spaces are closed

Definition
Let X be a topological space and $Y \subset X$ be any subset. Then

$$\overline{Y} = \bigcap_{Y\subset S} S \text{ closure of } Y \quad S \text{ closed in } Y \tag{75}$$
(Smallest closed set containing Y)
$$\text{Int}Y = \bigcup_{S\subset Y} S \text{ interior of } Y \quad S \text{ open in } Y \tag{76}$$
(Largest open set contained in Y)
$$\overline{Y} - \text{Int}Y = \partial Y \text{ boundary of } Y \tag{77}$$

Application: differential fields with boundary, Stoke's theorem.
To have a meaningful integral calculus on manifolds we need one more notion:
Definition
Let $(U_\alpha)_{\alpha\in A}$ be any open cover of X. A an arbitrary index set. Then an open cover $(V_\beta)_{\beta\in B}$ is called a refinement of $(U_\alpha)_{\alpha in A} \Leftrightarrow \forall V_\beta \exists U_\alpha \ni V_\beta \subset U_\alpha$.
One calls $(V_\beta)_{\beta\in B}$ locally finite if each $x \in X$ has an open nbh. $U \ni U \cap V_\beta \neq \varnothing$ for finitely many B.
X is said to be paracompact if every open cover has a locally finite refinement.
Theorem
Suppose that a topolgical space X is

- Hausdorff
- locally comapct (i.e. every $x \in X$ has a nbh. $U \ni \overline{U}$ is compact)
- a countable union of compact subsets

Then X is paracompact.
Remarks
We will apply this to manifolds which are typically paracompact in our applications. One can show that paracompact manifolds are second countable.
Warnings + Remarks

- If a net converges in a certeain topology $\Rightarrow$ converges in every weaker topology
- If a function is continuous in a certain topology $\Rightarrow$ continuous in every stroner one
- The Heine-Borel theorem that characterises compact sets
  (S is compact $\Leftrightarrow$ S is closed and bounded, boundness is a notion reserved for Banach spaces which are generalisations of Hilber spaces in which the metric is induced by a norm $\mathrm{d}(x,y) = ||x-y||$)
  is wrong in ∞-dim Banach spaces
- Notion of nets gives the topologist much more flexibility, sequences are in particular enough when X is second countable, in particular this is possible for paracompact manifolds

$\rightarrow$ Paracompactness for manifolds implies the existence of so called partition of unity which wmake it possible to import intergral calculus in $\mathbb{R}^n$ to a general manifold.

3 Manifolds finite dimensional

A manifold is a topological space which is locally homeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.
Definition

1. A topological space M is called a C^k manifold ($k = 1, 2, 3 \ldots, \infty$)

$$\exists \text{ family } (U_I, x_I)_{I \in \mathcal{I}} \quad \text{where } U_I \text{ from an open cover of } M \tag{78}$$
$$\text{and } x_I : U_I \to x_I (U_I) \subset \mathbb{R}^m \ , \ p \mapsto x_I(p) \tag{79}$$

   are homeomorphisms such that

$$\forall I, J \ \in \mathcal{I} \text{ with } U_I \cap U_J \neq \varnothing \tag{80}$$

   the maps

$$\phi_{IJ} : X_I \circ x_J^{-1} : x_J (U_I \cap U_J) \to x_I (U_I \cap U_J) \tag{81}$$

   is a C^k map between open subsets of $\mathbb{R}^m$ (k times continuously differentiable)

2. The U_I are called charts, the x_I are called local coordinates and the family $(U_I, x_I)_{I \in \mathcal{I}}$ is called an altas for M.
   Warning
   The homeomorphism x_I can vary drastically with the choice of $I \in \mathcal{I}$ amd therefore a global manifold can look very different from $\mathbb{R}^m$ when looked at globally. This happens when one does not have a global i.e. singel chart (U, x).

3. Two atlases $(U_I, x_I)_{I \in \mathcal{I}}, (V_J, y_J)_{J \in \mathcal{J}}$ are said to be compatible

$$\Leftrightarrow \text{The union } (w_\alpha, z_\alpha) \text{ with } \alpha \in \mathcal{I} \cup \mathcal{J} \tag{82}$$
$$w_\alpha = U_I \text{ if } \alpha = I, w_\alpha = V_J \text{ if } \alpha = J \tag{83}$$
$$z_\alpha = x_I, z_\alpha = y_J \tag{84}$$

   is again an atlas for M.
   One can show that compatibility of altasses is an equivalence relation. The corresponding equivalence classes are called the different C^k differentiable structures on M.

4. A topological space M is said to be a manifold with boundary provided that

$$x_I : U_I \to x_I(U_I) \subset \mathbb{R}^m_- \tag{85}$$
$$\mathbb{R}^m_- = \left\{ x \in \mathbb{R}^m ; x^1 \leq 0 \right\} \tag{86}$$

   a point $p \in M$ is said to be in the boundary ∂M of M

$$p \in U_I, x_I^1(p) = 0 \tag{87}$$

5. For C^k manifolds M, N a map $\psi : M \to N$ is called a C^k map

$$\Leftrightarrow \text{ If } M, N \text{ carry atlases } (U_I, x_I), (V_J, y_J) \ \wedge\ \psi(U_I) \cap V_J \neq \varnothing \tag{88}$$

$$\text{Then } \psi_{IJ} \coloneqq y_J \circ \psi \circ x_I^{-1}, x_I(U_I) \to y_J(V_J) \text{ is a } C^k\text{-map} \tag{89}$$

   If all the maps ψ_{IJ} are invertible and also C^k then ψ is called a C^k diffeomorphism. The C^k diffeos form a group under combination of maps and forming inverts. If $\psi_1, \psi_2, \ldots$ are C^k diffeos so are $\psi_1 \circ \psi_2$ and ψ_1^{-1}, ψ_2^{-1} exists, called the C^k diffeomorphism group $\mathrm{Diff}(M)^k$

6. If $N \subset M$ then we can induce a C^k differentiable structure on N as follows:
   If M has the atlas (U_I, x_I) then we can form

$$(V_I = N \cap U_I, y_I = x_I|_{V_I}) \tag{90}$$

$$\text{For } V_I \cap V_J \neq \varnothing \text{ , if } \tilde{\varphi}_{IJ} = y_J \circ y_I^{-1} \tag{91}$$

   has constant rank then this defines a C^k diff. structure on N.

7. Conversely, suppose that N is already an n-dimensional manifold and that $\psi : N \to M$ which is C^k in above sense, then ψ is called local immersion if $\forall q \in N \exists$ nbh. V of q in $N \ni \psi : V \to \psi(V)$ is injection. If moreover $\forall V \subset N$ open, $\psi(V)$ is open in the subspace (induced topology) of M, i.e. it is of the form $\psi(N) \cap N$ open in M, then ψ is called embedding. If $n = m - 1$, i.e. N has codimension 1, then N is called a hypersurface (initial value formulation, Cauchy problem in GR).

8. A manifold is called orientable $\exists$ atlas $\ni p \in U_I \cap U_J \neq \varnothing$ we have

$$\underbrace{\det \left(\frac{\partial X_I(P)}{\partial x_J(P)} \right)}_{Jacobian} > 0 \tag{92}$$

   i.e. the Jacobian has constant sign.
   If M is a manifold with boundary, then the atlas constructed above for ∂M is also orientable. It is called the induced orientation.

9. A manifold is called smooth if it is in C^∞

10. A manifold is called paracompact if paracompact as a topological space.

Theorem (Partition of unity $\to$ see integral calculus)
Given a locally finite atlas (U_I, x_I) of a paracompact C^k manifold there exist C^k functions e_I on M

1. $0 \leq e_I(P) \leq 1$

2. $\mathrm{supp}(e_I) \subset U_I$

3. $\sum_I e_I(P) = 1 \;\; \forall p \in M$

support of a function $\text{supp}(f) = \{p \in M, f(p) \neq 0\}$. Idea of proof: Import, using the local coordinates x_I, the construction of partitions of unity on $\mathbb{R}^m$. Important there are functions of the following type

- $[0,1] \ni x \mapsto e^{-1/x^2}$ is smooth even at $x = 0$

3.1 Differential calculus

- Functions
  A function $f : M \to \mathbb{C}$ is called smooth $\Leftrightarrow$ $f_I : x_I(U_I)_{\subset \mathbb{R}^m} \to \mathbb{C}$ is smooth $\forall I$ where $f_I(x) = f\left(x_I^{-1}(x)\right)$
  Note that $f : M \to \mathbb{C}$ is globally defined on M while f_I is only locally defined on $x_I(U_I)$. Still it is true that we have a coordinate independent presentation $f = f_I \circ x_I = f_j \circ x_J$ on $U_I \cap U_J$. One calls this property independence of the choice of a chart. Equivalentely we can think of f as the set of pairs $(f_I, x_I(U_I))$ f: coordinate independent, f_I: coordinate dependent (actual calculus are done here).

- Vector fields
  A smooth vector field is a derivation on $C^\infty(M)$ (smooth functions on M, $C^\infty(M)$ is a unital algebra under pointwise operations: e.g.
  - $(f+g)(p) = f(p) + g(p)$
  - $(f \cdot g) = f(p) \cdot g(p)$
  - $1(p) = 1 \; ; \; 1 \cdot f = f$

  i.e. the map $V : C^\infty(M) \to C^\infty(M)$ which is linear $v\left[\lambda f + \mu g\right] = \lambda v\left[f\right] + \mu v\left[g\right]$; $\lambda, \mu \in \mathbb{C}$, follows Leibniz rule: $v\left[f \cdot g\right] = v\left[f\right] \cdot g + f \cdot v\left[g\right]$. Annihilates constants: $v\left[1\right] = 0$.
  In particular we define for $f \in C^\infty(M)$, v a vector field

  $$((f \cdot v)\left[g\right])(p) := f(p)\,(v\left[g\right])(p) \tag{93}$$

  the space of smooth vector fields on M, denoted by $T^1(M)$ is a $C^\infty(M)$ module.

- Module
  A ring R is an associative unital algebra. A left R module or a vector space V is defined by an operation $:R \times V \to V$ called multiplication, $(f,v) \to f \cdot v$ which is compatible with the Ring operations
  - $(f+g) \cdot v = f \cdot v + g \cdot v$
  - $(f \cdot g) \cdot v = f \cdot (g \cdot v)$

  Here $V = T^1(M)$; $R = C^\infty(M)$
  Coordinate expressions subordinate to the choice of an atlas of M:

  $$(U_I, x_I) \tag{94}$$

  $$x_I^\mu : U_I \to \mathbb{R} \; ; \; p \mapsto x_I^\mu(p) \; , \; \mu = 1, 2, \ldots, m \tag{95}$$

We define corresponding vector fields ∂^I_μ on U_I

$$\left(\partial^I_\mu \left[x^\nu_I\right]\right)(p) := \delta^\nu_\mu \forall p \in M \tag{96}$$

Given $v \in T^1(M)$ arbitrary vector field we can build the following functions:

–

$$v^\mu_I : x_I(U_I) \to \mathbb{R} \tag{97}$$

$$v^\mu_I \left(X_I\left(p\right)\right) := \left(v\left[x^\mu_I\right]\right)(p) \forall p \in U_I \tag{98}$$

Claim:

$$v(p) = v^\mu_I \left(x_I(p)\right) \partial^I_\mu \forall p \in U_I \tag{99}$$

$$= \sum_{\mu=1}^{m} v^\mu_I \left(x_I(p)\right) \partial^I_\mu \tag{100}$$

Not only is this correct $\forall p \in U_I$ but moreover of $p \in U_I \cap U_J \neq \varnothing$ then in fact $v(p) = v^\mu_I \left(x_I(p)\right) \partial^I_\mu = v^\mu_\partial \left(x_J(p)\right) \partial^J_\mu$

4 Differential Calculus

The coordinate functions $x^\mu_I : U_I \to \mathbb{R}, p \to x^\mu_I(p)$, with $\mu = 1 \cdots m = dim(M)$, can be used to dfine vector fields over U_I as follows:

$$(\partial^I_\mu[x^\nu_I])(p) \equiv \delta^\nu_\mu \forall p \in U_I \tag{101}$$

Let for any vector field V a new function over U_I be defined by

$$v^\mu_I\left(x_I(p)\right) \equiv \left(v[x^\mu_I]\right)(p); v^\mu_I : x_I(U_I) \to \mathbb{R} \forall \mu = 1 \cdots m \tag{102}$$

It follows $v(p) = v^\mu_I(x_I(p))\partial^I_\mu(p)$

Proof:

We will always assume M to be paracompact. In particular, M is locally compact, i.e. every point $p \in M$ has a neighbourhood whose closure is compact. Now x_I is in particular a homeomorphism thus sends compact sets in M into compact sets in $\mathbb{R}^m$. This means that $f(q) = f_I(x_I(q))$ with $q \in K \subset M$, compact neighbourhood of p defines f_I as a function on the compact set $x_I(K) \subset \mathbb{R}^m$ Since f_I is smooth , it is in particular continuous. By the Weierstrass theorem, continuous functions on compact subsets of $\mathbb{R}^m$ can be approximated arbitrary well by polynomials,with regard to sup-norm.

$$\forall g \text{ on compact } C \subset \mathbb{R}^m, \epsilon > 0 \text{ we define } g_{pol} \text{ polynomial } \ni \sup_{x \in C} |g(x) - g_{pol}(x)| < \epsilon \tag{103}$$

choose $C = x_I(k)$ and $g = f_I$ we can apply the Weierstrass theorem to infer that $\exists$ a suitable polynomial $f_{I,pol} \forall \epsilon > 0$:

$$f_{I,pol} = \sum_{l=0}^{N} \underbrace{f_{I,pol\ \mu_1 \cdots \mu_l}}_{\text{constants}} x^{\mu_1} \cdots x^{\mu_l} \tag{104}$$

$$\Rightarrow f_{pol} = \sum_{l=0}^{N} \underbrace{f_{I,pol\ \mu_1 \cdots \mu_l}}_{\text{constants}} x^{\mu_1}_I \cdots x^{\mu_l}_I \tag{105}$$

is an approximation of f on M to precision ϵ .

$$v[f_{pol}] = \sum_{l=1}^{N} f_{I,pol\ \mu_1 \cdots \mu_l} \sum_{r=1}^{l} \underbrace{v[x^{\mu_r}_I]}_{=v^{\mu_r}_I} x^{\mu_1}_I \cdots \underbrace{x^{\hat{\mu}_r}_I}_{\text{missing}} \cdots x^{\mu_l}_I = (v^\mu_I \partial^I_\mu)[f_{pol}] \tag{106}$$

Thus $v = v^\mu_I \partial^I_\mu$ on polynomials and by taking limits $v = v^\mu_I \partial^\mu_I$ on all of $C^\infty(M)$

Remark:

Einstein summation conventions on $\mu, \nu, \cdots \in 1 \cdots m$ (tensor indices) but not on $I, J, \cdots \in \mathcal{I}$. Recall the transition functions $\varphi_{IJ} : x_I(U_I) \to x_0(U_J)\varphi_{IJ} = x_J \circ x_I^{-1}$:

$$v^\mu[x_J](p) = v^\mu_J(x_J(p)) = v^\mu[\varphi_{IJ}(x_I)] = \frac{\partial \varphi^\mu_{IJ}(x)}{\partial x^\nu}|_{x=x_I(p)} v[x^\nu_I] \tag{107}$$

On the one hand

$$(v[f])(p) = v^\mu_J(x_J(p)) \frac{\partial f_J(y)}{\partial y^\mu}|_{y=x_J(p)} = v^\nu_I(x_I(p)) \frac{\partial \varphi^\mu_{IJ}(x)}{\partial x^\nu}|_{x=x_I(p)} \frac{\partial f_J(y)}{\partial y^\mu}|_{y=x_J(p)} \tag{108}$$

$$\underbrace{=}_{\text{chain rule}} v^\nu_I(x_I(p)) \left[\frac{\partial}{\partial x^\nu} \underbrace{f_J(\varphi_{IJ}(x))}_{f_I(x)} \right]_{x=x_I(p)} \tag{109}$$

$$= v^\nu_I(x_I(p)) \frac{\partial f_I(x)}{\partial x^\nu}|_{x=x_I(p)} \quad \forall p \in U_I \cap U_J \text{ and } f_J = f \circ x_J^{-1} \tag{110}$$

We distinguish between globally defined objects like $f, v, \cdots$ and local coordinate expressions like $f_I, v^\mu_I, \cdots$ and must make sure that these local expressions are consistent on intersections $U_I \cap U_J \neq \varnothing$ in other words they must be compatible with the coordinate transformations φ_{IJ} which <u>define</u> the differentiable structure of the manifold.

This theme will now repeat all over the place when we develop <u>tensor analysis</u>:

Definition:

One-forms. Let $T^1(M)$ be the set of all smooth vectorfields on M. Then a smooth one-form is a map $\omega : T^1(M) \to C^\infty(M); v \to \omega[v]$, s.t.

$$\omega[fu + gv] = f\omega[u] + g\omega[v] \forall f, g \in C^\infty(M) \text{ and } u, v \in T^1(M) \tag{111}$$

linear and compatible with the module structure of $T^1(M)$. For $f \in C^\infty(M)$ we define df one-form by $(df)[v] = v[f] \forall v \in T^1(M)$.

Application to the coordinate functions:

$$(dx^\mu_I)[v] = v[x^\mu_I] = v^\mu_I, \text{ In particular } dx^\mu_I[\partial^I_\nu] = \partial^I_\nu[x_I] = \delta^\nu_\mu \tag{112}$$

Let $\omega^I_\mu(x_I(p)) \equiv \left(\omega[\partial^I_\mu]\right)(p)$ functions ω^I_μ on $x_I(U_I)$

<u>Claim:</u> $\omega(p) = \omega^I_\mu(x_I(p)) dx^\mu_I(p)$ and is independent of the choice of I (ex. analogous to the corresponding claim for $v = v^\mu_I \partial^I_\mu$) In particular:

$$(df)(p) = \frac{\partial f_I}{\partial x^\mu}|_{x=x_I(p)} dx^\mu_I(p) \tag{113}$$

Tensor fields: Notation: $T^0_0(M) = c^\infty(M)$

$$T^1_0(M) \equiv T^1(M) \text{ smooth vector fields} \tag{114}$$

$$T^0_1(M) \equiv T_1(M) \text{ smooth one forms} \tag{115}$$

A smooth tensor field t of type $(a,b) a,b \in \mathbb{N}_0$ (a-times contravariant and b-times covariant) is a multilinear functional.

$$t : \Pi_{r=1}^{a} T_1(M) \times \Pi_{s=1}^{b} T^1(M) \to C^\infty(M); \tag{116}$$
$$(\omega_1 \cdots \omega_a) \times (v_1 \cdots v_b) \to t[\omega_1 \cdots \omega_a; v_1 \cdots v_b] \tag{117}$$

multilinear means linear in every entry

$$t[\omega_1 \cdots, \omega_c^{(1)} + \omega_c^{(2)} \cdots \omega_a; v_1 \cdots v_b] = \tag{118}$$
$$t[\omega_1 \cdots, \omega_c^{(1)} \cdots \omega_a; v_1 \cdots v_b] + t[\omega_1 \cdots, \omega_c^{(2)} \cdots \omega_a; v_1 \cdots v_b] \quad \forall 1 \le c \le a \tag{119}$$
$$\text{and } t[\omega_1 \cdots \omega_a; v_1 \cdots, v_d^{(1)} + v_d^{(2)} \cdots v_b] = \tag{120}$$
$$t[\omega_1 \cdots \omega_a; v_1 \cdots, v_d^{(1)} \cdots v_b] + t[\omega_1 \cdots \omega_a; v_1 \cdots, v_d^{(2)} \cdots v_b] \quad \forall 1 \le d \le b \tag{121}$$

With the definition of the functions

$$t_{I\ \nu_1 \cdots \nu_b}^{\mu_1 \cdots \mu_a}(x_I(p)) \equiv t[dx_I^{\mu_1} \cdots dx_I^{\mu_a}; \partial_{\nu_1}^I \cdots \partial_{\nu_b}^I](p) \tag{122}$$

we have the following formula

$$t(p) = t_{I\ \nu_1 \cdots \nu_b}^{\mu_1 \cdots \mu_a}(x_I(p)) \partial_{\mu_1}^I \otimes \cdots \otimes \partial_{\mu_a}^I \otimes dx_I^{\nu_1} \otimes \cdots \otimes dx_I^{\nu_b} \tag{123}$$

Definition:

Tensor product is defined as follows:

$$(\omega_1 \otimes \omega_2)(v_1, v_2) = \omega_1(v_1)\omega_2(v_2) \tag{124}$$
$$(v_1 \otimes v_2)(\omega_1, \omega_2) = \omega_1(v_1)\omega_2(v_2) \text{ etc.} \tag{125}$$

Indep. of the coordinate chart.

We can go even further: For $t \in T_b^a(M), t' \in T_d^c(M)$ we define $t \otimes t' \in T_{b+d}^{a+c}(M)$ by

$$(t \otimes t')[\omega_1 \cdots \omega_{a+c}; v_1 \cdots v_{b+d}] = t[\omega_1 \cdots \omega_a; v_1 \cdots v_b] \cdot t'[\omega_{a+1} \cdots \omega_{a+d}; v_{b+1} \cdots v_{b+d}] \tag{126}$$

Tensor field algebra $T(M)$ over M as

$$T(M) = \oplus_{a,b=0}^{\infty} T_b^a(M) \tag{127}$$
$$T(M) \ni t = \oplus_{1,b=0}^{\infty}(t_b^a) = (t_0^0, t_1^0, t_0^1, \cdots) \tag{128}$$

with $t_b^a \neq 0$ for finitely many tuples (a,b). This is an algebra with multiplication given by the tensor product:

$$t \otimes t' = \oplus_{a,b=0}^{\infty}[\sum_{a'=0}^{a} \sum_{b'=0}^{b} t_{b'}^{a'} \otimes (t')_{b-b'}^{a-a'}] \tag{129}$$

Important conclusion: To understand tensor analysis over M it is sufficient to know $\mathbb{C}^\infty(M)$ and $T^1(M)$ because from $\omega = \omega^I_\mu dx^\mu_I$ we see that any $\omega \in T_1(M)$ is a linear combination of 1-forms of the form $gdf, g, f \in C^\infty(M)$. Everything else just is done using tensor products.

Abstract index notation

t as a tensor field is globally defined and has local coordinate expressions $t^{\mu_1 \cdots \mu_a}_{I\nu_1 \cdots \nu_b}$ for its components with regard to the bases ∂^I_μ, dx^μ_I over U_I which are such that $t = t^{\mu_1 \cdots \mu_a}_{I\ \nu_1 \cdots \nu_b} \partial^I_{\mu_1} \otimes \cdots \partial^I_{\mu_a} \otimes dx^{\nu_1}_I \cdots \otimes dx^{\nu_b}_I$ is independent of I, ∂ when $U_I \cap U_\partial \neq \varnothing$. Instead of saying $t \in T^a_b(M)$ with local expressions $+_I$ we just talk about $t^{\mu_1 \cdots \mu_a}_{\nu_1 \cdots \nu_b}$ just dropping index I . We use this notation to indicate the type of tensor and wich type of manipulations we perform with it.

Example:

Contraction with a vector v of a tensorfield $t \in T^a_b(M), b \neq 0$ in entry $1 \leq k \leq b$ could be denoted by $i^k_v t \in T^a_{b-1}$ and defined in abstract index notation by

$$(i^k_v t)^{\mu_1 \cdots \mu_a}_{\nu_1 \cdots \nu_{b-1}} = t^{\mu_1 \cdots \mu_a}_{\nu_1 \cdots \nu_{k-1} \nu \nu_k \nu_{b-1}} v^\nu \tag{130}$$

For every operation we would need to introduce new symbols but the abstract index notation does not need that and is much shorter.

n-forms:

$$\Lambda_n(M) = \text{ totally skew n-times covariant tensorfields} \tag{131}$$
$$\Lambda_n(M) \subset T^0_n(M) \tag{132}$$
$$\omega \in \Lambda_n(M) \Leftrightarrow \omega[v_1, \cdots v_n] = \text{sgn}(\pi)\omega[v_{\pi(1)} \cdots v_{\pi(n)}] \forall \pi \in S_n \tag{133}$$
$$: \text{ symmetric group in } n \text{ symbold} \tag{134}$$
$$\Lambda_n(M) = 0 \forall n > M \tag{135}$$
$$\Lambda_0(M) \equiv C^\infty(M); \Lambda(M) = \oplus^m_{n=0} \Lambda_n(M) \tag{136}$$

$\Lambda(M)$ defines grassman algebra.

To make this an algebra we need a special tensor product, called the exteriour product.

$$\Lambda : \Lambda_k(M) \otimes \Lambda_l(M) \to \Lambda_{k+l}(M) : (\omega, \sigma) \to \omega \wedge \omega \tag{137}$$
$$(\omega \wedge \sigma)[v_1, \cdots, v_{k+l}] \equiv \frac{1}{k!l!} \sum_{\pi \in S_{k+l}} \text{sgn}(\pi) \omega[v_{\pi(1)} \cdots v_{\pi(k)}] \sigma[v_{\pi(k+1)} \cdots v_{\pi(k+l)}] \tag{138}$$

by inspection totally skew again, globally defined.

Operations in $\Lambda(M)$ exterior dericative:

$$d : \Lambda_n(M) \to \Lambda_{n+1}(M); \omega \to d\omega \tag{139}$$
$$d\omega(v_0 \cdots v_n) \equiv \sum^n_{k=0} (-1)^k v_k[\omega[v_0 \cdots \underbrace{\hat{v_k}}_{\text{missing}} \cdots v_n]] + \sum_{0 \leq k \leq l \leq n} (-1)^{k+l} \omega[[v_k, v_l] v_0 \cdots \hat{v}_k \hat{v}_l \cdots v_n] \tag{140}$$

(where the second term is a correction term that avoids derivatives of the vectorfield) where $[v_k, v_l]$ is the commutator of vector fields which is defined as a new vector field:

$$([v_k, v_l])[f] = v_k[v_l[f]] - v_l[v_k[f \tag{141}$$

$$([v_k, v_l])[fg] = v_k[v_l[f]g + fv_l[g]] - v_l[v_k[f]g + fv_k[g]] = ([v_k, v_l])[f]g + f([v_k, v_l])[g] \tag{142}$$

Definition:

Interior product with a vector field

$$i_v : \Lambda_n(M) \to \Lambda_{n-1}(M); (i_v\omega)[v_1 \cdots v_{n-1}] \equiv \omega[vv_1 \cdots v_{n-1}] \tag{143}$$

To memorise these formulas, the following notation is used in practice:

Definition:

Let us define total (anti-)symmetrisation in n symbols.

$$T[I_1 \cdots I_n] \equiv \frac{1}{n!} \sum_{\pi \in S_n} \operatorname{sgn}(\pi) T_{I_{\pi(1)} \cdots I_{\pi(n)}} \tag{144}$$

$$T(I_1 \cdots I_n) \equiv \frac{1}{n!} \sum_{\pi \in S_n} T_{I_{\pi(1)} \cdots I_{\pi(n)}} \tag{145}$$

Independent operation: Its' square reproduces the original operation.

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \equiv n! dx^{[\mu_1} \otimes \cdots \otimes dx^{\mu_n]} \tag{146}$$

$$= \sum_{\pi \in S_n} \operatorname{sgn}(\pi) dx^{\mu_{\pi(1)}} \otimes \cdots \otimes dx^{\mu_{\pi(n)}} \tag{147}$$

$$\Rightarrow \omega = \omega_{\mu_1 \cdots \mu_n} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_n} = \omega_{[\mu_1 \cdots \mu_n]} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_n} \tag{148}$$

$$\underbrace{=}_{\text{relabel}} \omega_{\mu_1 \cdots \mu_n} dx^{[\mu_1} \otimes \cdots \otimes dx^{\mu_n]} = \frac{1}{n!} \omega_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{149}$$

Exercise:

1. $\omega \wedge \sigma = \frac{1}{k!l!} \omega_{\mu_1 \cdots \mu_k} \sigma_{\mu_{k+1} \cdots \mu_{k+l}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{k+l}}$
2. $d\omega = \frac{1}{k!} (\frac{\partial}{\partial x^{\mu_0}} \omega_{\mu_1 \cdots \mu_k}) dx^{\mu_0} \wedge \cdots \wedge dx^{\mu_k}$
3. $i_v \omega = \frac{1}{(k-1)!} v^{\mu} \omega_{\mu \mu_1 \cdots \mu_{k-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{k-1}}$

4.0.1 Properties

- Not commutative

$$\omega \wedge \sigma = \frac{1}{k!l!}\omega_{\mu_1\cdots\mu_k}\sigma_{\mu_{k+1}\cdots\mu_{k+l}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{k+l}} \tag{150}$$

  we can pull the last l indices in front of the first k $\rightarrow$ affords k and l transpositions

$$= \frac{1}{l!k!}(-1)^{kl}\sigma_{\mu_{k+1}\cdots\mu_{k+l}}\omega_{\mu_1\cdots\mu_k}dx^{\mu_{k+1}}\wedge\cdots\wedge dx^{\mu_{k+l}}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k} \tag{151}$$

$$\underbrace{=}_{relabel}(-1)^{kl}\frac{1}{k!l!}\sigma_{\mu_1\cdots\mu_l}\omega_{\mu_{k+1}\cdots\mu_{k+l}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{k+l}} \tag{152}$$

$$= (-1)^{kl}\sigma\wedge\omega \tag{153}$$

- But with the same manipulations associative

$$(\omega\wedge\sigma)\wedge\tau = \omega\wedge(\sigma\wedge\tau) \tag{154}$$

- $d^2 = 0$, for ω an n-form:

$$d\omega = \frac{1}{n!}\left(\partial_{\mu_0}\omega_{\mu_1\cdots\mu_n}\right)dx^{\mu_0}\wedge\cdots\wedge dx^{\mu_n} \tag{155}$$

$$= \frac{1}{(n+1)!}\underbrace{\left((n+1)\partial_{\mu_0}\omega_{\mu_1\cdots\mu_n}\right)}_{\tilde{\omega}_{\mu_1\cdots\mu_{n+1}}}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{n+1}} \tag{156}$$

$$d^2\omega = \frac{1}{(n+1)!}\left(\partial_{\mu_0}\tilde{\omega}_{\mu_1\cdots\mu_{n+1}}\right)dx^{\mu_0}\wedge\cdots\wedge dx^{\mu_{n+1}} \tag{157}$$

$$= \frac{1}{n!}\underbrace{\left(\frac{\partial}{\delta x^{\mu_0}\partial x^{\mu_1}}\omega_{\mu_1\cdots\mu_{n+1}}\right)}_{=0}dx^{\mu_0}\wedge\cdots\wedge dx^{\mu_{n+1}} = 0 \tag{158}$$

- $d(\omega\wedge\sigma) = d\omega\wedge\sigma + (-1)^{deg(\omega)}\omega\wedge d\sigma$ with $deg(\omega) = k$ the degree of ω if ω is a k-form
- $i_v^2 = 0$
- $i_v df = df[v] = v[f] = v^\mu(\partial_\mu f)$ Def. of d on k-forms consistent with earlier def. of df for $f\in C^\infty(M)$
- $i_v(\omega\wedge\sigma) = (i_v\omega)\wedge\sigma + (-1)^{deg(\omega)}\omega\wedge(i_v\sigma)$

Definition:
A k-form ω is called closed if and only if (iff) $d\omega = 0$. It is called exact iff $\exists(k-1)$-form $\sigma s.t. d\sigma = \omega$

As $d^2 = 0$ every exact k-form is closed but not vice versa. It is always locally true (Lemma of Poincaré), also holds on a general manifold. Globally in general false and the modul space of closed modulo exact k-forms on M is a vectorspace $C_k(M)$ called the k-th cohomolgy "group" of M. It encodes important information about the topology of M. We will come back to this topic later.

4.1 Tensor Transformation laws

4.1.1 Passive diffeomorphisms

Recall that over U_I we have the following coordinate dependent expressions for $t \in T^a{}_b(M); t(p) = (t_I)^{\mu_1\cdots\mu_a}_{\nu_1\cdots\nu_b}(x_I(p))\partial^I_{\mu_1}(p) \otimes \cdots \otimes \partial^I_{\mu_a}(p) \otimes dx^{\nu_1}_I(p) \otimes \cdots \otimes dx^{\nu_b}_I(p)$ For $p \in U_I \cap U_J \neq \varnothing$ we can change coordinates

$$\partial^J_\mu = \frac{\partial x^\nu_I}{\partial x^\mu_J}\partial^I_\nu \underbrace{=}_{\varphi_{JI}\circ\varphi_{IJ}=\infty} \frac{\partial \varphi^{-1}_{IJ}}{\partial x^\mu_J}\partial^I_\nu \tag{159}$$

$$dx^\mu_J = \frac{\partial x^\mu_J}{\partial x^\nu_I}dx^\nu_I = \frac{\partial \varphi^\mu_{IJ}}{\partial x^\nu_I}dx^\nu_I \tag{160}$$

$(x_J = \varphi_{IJ} \circ x_I)$ Expand the ∂ basis ∂^J_μ and J-cobasis dx^μ_J into I-basis and cobasis and compare coefficients..

$$t_I{}^{\mu_1\cdots\mu_a}_{\nu_1\cdots\nu_b}(x_I(p)) = t_J{}^{\mu'_1\cdots\mu'_a}_{\nu'_1\cdots\nu'_b}(\varphi_{IJ}(x_I(p)))[\Pi^a_{k=1}\left(\frac{\partial(\varphi^{-1}_{IJ})^{\mu_k}}{\partial x^{\mu'_k}_J}\right)][\Pi^b_{l=1}\left(\frac{\partial\varphi^{\nu'_l}_{IJ}}{\partial x^{\nu'_l}_I}\right)] \equiv (\varphi^*_{IJ}t_J)^{\mu_1\cdots\mu_a}_{\nu_1\cdots\nu_b}(x_I(p)) \tag{161}$$

φ^* is called the pull-back transformation in this case acting on t_J and we find that a tensor field on M with local coordinate expressions t_I is globally def. iff $\varphi^*_{IJ}t_J = t_I\ \forall I, J$ s.t. $U_I \cap U_J \neq \varnothing$. In general for a diffeomorphism φ on a subset of $\mathbb{R}^m$ we call $\varphi^* t$ as above with $\varphi_{IJ} \to \varphi, t_I \to t$ the pull-back of t and $\varphi_* t = (\varphi^{-1})^* t$ its push-forward. Reasons for the name "passive": The change of coordinates $x_I \to x_J$ does not change the point $p \in M$, we just give it a new name. By contrast, when we alse change $p \in M$ we get technically very similar formulas, with a new meaning of the transformation φ.

4.1.2 Active Diffeomorphisms

$\Phi : M \to M; \varphi^{-1}\exists$, both smooth we may define the pull-back of c^∞-functions by $(\Phi^* f)(p) \equiv f(\Phi(p))$ in a natural way. For a vector field $v \in T^1(M)$ we may define in a natural way its push-forward: $((\varphi_* v)[f])(\Phi(p)) \equiv v[\Phi^* f](p)$. Reasons for terminology

- $\Phi^* f$ is a new function which is just given in terms of f but "pulled back" to the old point p
- $\Phi_* v$ is a new vector field which is just given in terms of v but "pushed forward" to the point $\Phi(p)$

For $\omega \in T_1(M)$ we define its pull-back by $\left(\left(\Phi^*\omega\right)[v]\right)(p) = \left(\omega\left[\Phi_* v\right]\right)\left(\Phi(p)\right)$.

If we restrict to tensors of type $T^a_0(M)$ or $T^0_b(M)$ then it is enough to deal with Φ because we can just use duality again.

$$\left(\Phi_* t\right)[\omega_1, \cdots, \omega_a] = t(\Phi^*\omega_1, \cdots, \Phi^*\omega_a); t \in T^a_0(M) \tag{162}$$

$$\left(\Phi_* t\right)[v_1, \cdots, v_b] = t(\Phi^* v_1, \cdots, \Phi^* v_b); t \in T^0_b(M) \tag{163}$$

$$\tag{164}$$

For mixed tensors of type $T^a_b(M)$ with $a, b \neq 0$ we also need Φ^{-1}. This has the following application: so far Φ: $M \to N$ could have been any smooth map between manifolds not necessarily invertible. In particular, M could be a submanifold of N and given a k-form ω on N, $\Phi^*\omega$ would be a k-form on $M \to$ Integration on submanifolds.

Destinction between passive and active deffeomorphism:

- passive: From using different coordinates for same points $p \in M$
- active: From changing points $p \to \psi(p)$ in M

For passive diffeos. we discovered the trafo laws by writing $t(p)$ in terms of the bases

$$\partial^I_\mu, \mathrm{d}x^\mu_I, \partial^J_\mu, \mathrm{d}x^\mu_J \text{ over } U_I \cap U_J \neq \varnothing \tag{165}$$

For active ones we defined $(\psi * f)(p) = f(\psi(p))$ pull-back of $f \in C^\infty(M)$

- push-forward of $v \in T^1_0(M)$: $((\psi_* v)[f])(\psi(p)) := (v[\psi^* f])(p)$
- pull-back of $w \in T^0_1(M)$: $((\psi^* w)[v])(p) := (w[\psi_* v])(\psi(p))$

Likewise for $t \in T^a_0(M)$ may define

$$(\psi_* t)[w_1, \ldots, w_a] := t(\psi^* w_1, \ldots, \psi^* w_a) \tag{166}$$

For $t \in T^0_b(M)$ we may define $(\psi^* t)(v_1, \ldots, v_b) := t(\psi_* v_1, \ldots, v_b)$
For mixed tensors we want to define ψ^*, ψ_* in such a way that the left hand side and the right hand side of the transormation formula we only have objects defined at the same point. i. e. the <u>new</u> tensors $\psi^* t$ or $\psi_* t$ at some point should be defined in terms of the <u>old</u> tensor t at some, but the <u>same</u> point.

$$((\psi^* t)[w_1, \ldots, w_a, v_1, \ldots, v_b])\, p := \left(t\left[(\psi^{-1})^* w_1, \ldots, (\psi^{-1})^* w_a, \psi_* v_1, \ldots, \psi_* v_b\right]\right)(\psi(p)) \tag{167}$$

<u>Similarly</u>

$$((\psi_* t)[w_1, \ldots, w_a, v_1, \ldots, v_b])(\psi(p)) := \left(t\left[\psi^* w_1, \ldots, \psi^* w_a, (\psi^{-1})_* v_1, \ldots, (\psi^{-1})_* v_b\right]\right)(p) \tag{168}$$

natural extensions of the T^a_0, T^0_b cases which satisfy the above criterion. Note that for T^a_0, T^0_b ψ^{-1} is not needed. It is only neede for the mixed tensor types.

Exercise:
Suppose $\psi, \psi' \in \mathrm{Diff}(M)$ two diffeomorphisms of M. show

- $\psi^* \circ (\psi')^* = (\psi' \circ \psi)^*$
- $\psi_* \circ \psi'_* = (\psi \circ \psi')_*$
- $\psi_* = (\psi^*)^{-1} = (\psi^{-1})^*$

Definition:

Coordinate expressions for active Diffeos $\psi \in \mathrm{Diff}(M)$
Let (U_I, x_I) atlas for M. Consider the equivalent atlas $V_I = \psi^{-1}(U_I), v_I = x_I \circ \psi$
Suppose $p \in U_I, \psi(p) \in U_J$
$\Rightarrow p \in \psi^{-1}(U_J) = V_J$ and $y_J = x_J \circ \psi$ are local coordinates on V_J by definition. Then $\psi_{IJ} := y_J \circ x_I^{-1} : x_I(U_I \cap V_J) \to y_J(U_I \cap V_J)$ and we have

$$(\psi^* f)(p) = (\psi^* f)_I(x_I(p)) = f(\psi(p)) \tag{169}$$
$$= f_J(x_J(\psi(p))) = f_J(\underbrace{x_J \circ \psi \circ x_I^{-1}}_{\psi_{IJ}}(x_I(p))) \tag{170}$$
$$= f_J(\psi_{IJ}(x_I(p))) \tag{171}$$

compare with the formula $\psi_{IJ}^* x_J$ for passive diffeo.

Similarly, using the definitions one finds

$$((\psi_* v)[f])(\psi(p)) = (\psi_* v)_J^\mu (x_J(\psi(p)))(\partial_\mu^J f_J)(x_J(\psi(p))) \tag{172}$$
$$= v_I^\nu(x_I(p)) \left.\frac{\partial \psi_{IJ}(x)}{\partial x^\nu}\right|_{x=x_I(p)} (\partial_\mu^J f_J)(\psi_{IJ}(x_I(p))) \tag{173}$$
$$(\psi_* v)_J^\mu(\psi_{IJ}(x)) = \frac{\partial \psi_{IJ}^\mu(x)}{\partial x^\nu} v_I^\nu(x) \tag{174}$$

exatcly the same as before just that $\varphi_{IJ} \leftrightarrow \psi_{IJ}$, likewise:

$$(\psi^* w)_\mu^I(x) = w_\mu^J(\psi_{IJ}(x)) \frac{\partial \psi_{IJ}^\nu(x)}{\partial x^\mu} \tag{175}$$

again: exactly the same as before.
$\Rightarrow$ the general coord. expressions for $\psi^* t, \psi_* t$ are obtained by exchanging $\varphi_{IJ} \leftrightarrow \psi_{IJ}$.

5 Integral curves and lie derivatives

Definition:

1. A map $c : [a,b] \subset \mathbb{R} \to M, t \mapsto c(t)$ is called a smooth curve in $M \leftrightarrow t \to x_I(c(t)) : [a,b] \mapsto x_I(U_I) \subset \mathbb{R}^m$ is smooth $\forall I \in \tilde{I}$ for which $c([a,b]) \cap U_I \neq \varnothing$.

2. The tangent vector field T_c of a smooth curve along the curve is defined by the condition $(T_c[f])(c(t)) = \frac{\mathrm{d}}{\mathrm{d}t} f(c(t)) \forall f \in C^\infty(M)$

$$= \left.\frac{\partial f_I(x)}{\partial x^\mu}\right|_{x=x_I(c(t))} \frac{\mathrm{d}x_I(c(t))}{\mathrm{d}t} \tag{176}$$

in terms of local coordinates.

$$\Rightarrow (T_c)_I^\mu(x_I(p))\big|_{p=c(t)} = \frac{\mathrm{d}}{\mathrm{d}t} x_I^\mu(c(t)) \tag{177}$$

This connection between curves and tangent vector fields is exactly what we are used to from the corresponding notions in $\mathbb{R}^m$.

3. $c \to T_c$ defines a tangent vector field along c. Conversely, given any vector field $v \in T_0^1(M)$ we can define the following integral curves through points $p \in M$. $t \mapsto c_P^v(t)$ is called the integral curve through $p \in M$

   a) $c_p^v(0) = p$

   b) $T_{c_p^v} = v$ on c_p^v

   i.e. the integral curve of v through p is such that its tangent vector field agrees with v on this curve.
   In terms of local coordinates this translates into the following system of ordinary first order differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} x_I^\mu(c_p^v(t)) = v_I^\mu(x_I(c_p^v(t))), x_I^\mu(c_p^v(0)) = x_I^\mu(p) \tag{178}$$

   if we just abbreviate $x_I^\mu(c_p^v(t)) =: x^\mu(t), x_I^\mu(c_p^v(0)) = x_o^\mu$

$$\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(t) = v_I^\mu(x(t)), x^\mu(0) = x_0^\mu \tag{179}$$

   Such a system as a unique and maximal solution defined on some interval $[a, b]$ containing zero. The maximal solution is called the integral curve.
   The integral curve $c_p^v \exists \forall p \in M$ and the set of integral curves $\left\{c_p^v : p \in M\right\}$ is called the flow of v.
   Application: Two magnets on a table: scatter iron dust on it and since the iron is a ferromagnet the iron particles glue together in chains that want to align along the magnetic field lines. These are examples of such integral curves.

4. Given a vector field v and its flow we may define a 1-parameter set of local active diffeos $\psi_t^v, \psi_t^v(p) := c_p^v(t)$ is the point to which p is mapped along the integral curve. Given $p_0 \in M$ we consider a neighbourhood U of p_0 and a sufficiently small interval $[a, b]$ around zero such that the integral curves in U do not cross. What is the nature of this family $t \to \psi_t^v$ of diffeomorphisms?
   Claim $\psi_s^v \circ \psi_t^v = \psi_{s+t}^v$ as long as s, t are sufficiently small.

   Proof:

$$t \to (\psi_I^v \circ \psi_s^v)(p) = \psi_t^v(\psi_s^v(p)) = c^v_{\psi_s^v(p)}(t) \tag{180}$$

   is the integral curve of v through $\psi_s^v(p)$. On the other hand $t \to \psi_{s+t}^v(p) = c_p^v(t+s) = \tilde{c}(t)$ satisfies $\frac{\mathrm{d}}{\mathrm{d}t} x^\mu(\tilde{c}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} x^\mu(c_p^v(\underbrace{t+s}_{r})) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{r=s+t} x^\mu(c_p^v(r)) v^\mu(x(c_p^v(s+t)))$
   $\Rightarrow \tilde{c}(t)$ is the integral curve of v through $\tilde{c}(0) = c_p^v(s) = \psi_s^v(p)$. By uniqueness of solutions of systems of ODE's we have proofed that claim.

5. The Lie derivative of a tensor $t \in t^a_b(m)$ is defined by

$$\mathcal{L}(t)(p) \coloneqq \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} [(\psi^v_s)^* t]\,(p) \tag{181}$$

This defines a new tensor field $\mathcal{L}_v t$ because it is the differential quotient of the tensor field $\psi^v_s * t$.

Local Coordinates expressions: Recall (in abstract index notation)

$$[(\psi^v_s)^* t]\,(x) = t^{\mu_1,\dots,\mu_a}\nu_1,\dots,\nu_b(\psi(x)) \prod_{k=1}^{a} \left.\frac{\partial(\psi^{-1}_s)^{\mu_k}(y)}{\partial y^{\mu_k}}\right|_{y=\psi_s(x)} \prod_{l=1}^{b} \frac{\partial \psi^{\mu_l}_s(x)}{\partial x^{\nu_l}} \tag{182}$$

By definition: $\frac{\mathrm{d}}{\mathrm{d}s}\psi^\mu_s(x) = \frac{\mathrm{d}}{\mathrm{d}s}x^\mu(\psi^\nu_s(p)) = v^\mu(x(\psi^v_s(p)))$

$$\Rightarrow \left.\frac{\mathrm{d}}{\mathrm{d}s}\psi^\mu_s(x)\right|_{x=0} = v^\mu(x) \tag{183}$$

On the other hand due to $\psi^v_s \circ \psi^v_s = \mathrm{id} = \psi^v_s \circ (\psi^v_s)^{-1}$ we have $(\psi^v_s)^{-1} = \psi^v_s$ thus $\frac{\mathrm{d}}{\mathrm{d}s}(\psi^{-1}_s)^\mu\big|_{s=0} = -v^\mu(x)$.

Since the derivatives $\frac{\partial}{\partial s}$, $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial y^\mu}$ commutate for instance $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} \frac{\partial \psi^\mu_s(x)}{\partial x^\nu} = \frac{\partial v^\mu(x)}{\partial x^\nu}$ etc.

$$(\mathcal{L}_v t)^{\mu_1,\dots,\mu_a}_{\nu_1,\dots,\nu_b}(x) = v^\rho(x)\left[\frac{\partial}{\partial x^\rho} t^{\mu_1,\dots,\mu_a}_{\nu_1,\dots,\nu_b}(x)\right] - \sum_{k=1}^{a} \frac{\partial v^{\mu_k}(x)}{\partial x^{\mu_k}} t^{\mu_1,\dots,\hat{\mu}_k,\mu'_k,\dots,\mu_a}_{\nu_1,\dots,\nu_b})(x) + \sum_{l=1}^{b} \frac{\partial v^{\nu'_l}(x)}{\partial x^{\nu_l}} t^{\mu_1,\dots,\nu_a}_{\nu_1,\dots,\hat{\nu}_l,\nu'_l,\dots,\nu_b}(x)$$

Exercise:

Show directly that this transforms as an element of $T^a_b(M)$ given the known transformation formulas for v and t.

5.1 Applications of the Lie derivative

A rotationally invariant function on $\mathbb{R}^3$ with rotations around the origin are simply functions which only depend on the radial coordinate. Because f is invariant $\Leftrightarrow f(k) = f(Rx)$ where R is a rotation $R \in \mathrm{SO}(3)$. Consider 1-parameter subgroups $s \to R_s \in \mathrm{SO}(3)$ then of course f invariant if $\psi^*_s f = f$ where $\psi_s(x) = R_s x$ is a diffeo of $\mathbb{R}^3$. This no generalizes to manifolds and from functions to tensor fields:

$$\mathcal{L}_v f = 0 \text{ where } v \text{ is a vector field tangent to spheres} \tag{184}$$

$\Rightarrow$ rotationally invariant metric tensors, special solutions of Einstein equations, whose unique solution is given by the Schwarzschild solution (Birkhoff's theorem)

6 Derivations on the Grassman Algebra

$$\Lambda(M) = \bigoplus_{p=0}^{m} \Lambda_p(M) \tag{185}$$

There is a nice interplay between the operations d (exterior derivative), i_v (interior product) and $\mathcal{L}_v$ (lie derivative). As a preparation:

Lemma:
$\mathrm{d}\psi* = \psi * \mathrm{d}$

Proof:
For $f \in C^\infty(M)$ we have

$$((\psi^* \,\mathrm{d}f)[v])(p) = (\mathrm{d}f[\psi_* v])(\psi(p)) \tag{186}$$

$$= ((\psi_* v)[v])(\psi(p)) = v[\psi^* f](p) = (\mathrm{d}(\psi^* f)[v])(p) \tag{187}$$

Thus the statement holds on $\Lambda_0(M) = C^\infty(M)$. This turns out to be sufficient because any n-form is a linear combination of n-forms of the form $w = f_0 \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ ($w = \frac{1}{n!} w_\mu \cdots w^{\mu_n} \,\mathrm{d}x^{\mu_n} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$). ψ^* is linear so we may reduce the proof to these.

$$\begin{aligned}
\mathrm{d}(\psi^*[f_0 \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n]) &= \mathrm{d}((\psi^* f_0)(\psi^* \,\mathrm{d}f_1) \wedge \cdots \wedge (\psi^* \,\mathrm{d}f_n)) \\
&= \mathrm{d}((\psi^* f_0) \,\mathrm{d}(\psi^* f_1) \wedge \cdots \wedge \mathrm{d}(\psi^* f_n)) \\
&= \mathrm{d}(\psi^* f_0) \wedge \mathrm{d}(\psi^* f_1) \wedge \cdots \wedge \mathrm{d}(\psi^* f_n) \\
&= \psi^*(\mathrm{d}f_0 \wedge \cdots \wedge \mathrm{d}f_n) \\
&= \psi^* \,\mathrm{d}(f_0 \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n)
\end{aligned}$$

Definition:

1. A linear operation $D \; \Lambda_p(M) \to \Lambda_{p+d}(M)$ is called to be of degree $d \in \mathbb{Z}$
2. If d is even or odd and D satisfies the Leibniz or Anti-Leibniz rule $D(w \wedge \sigma) = (Dw) \wedge \sigma + w \wedge (D\sigma)$ or $D(w \wedge \sigma) = (Dw) \wedge \sigma + (-1)^{\deg(w)} w \wedge (DG)$ then it is called a derivation of or Anti-derivation respectively. More compactly $D(w \wedge \sigma) = (Dw) \wedge \sigma + (-1)^{d \deg(w)} w \wedge (D\sigma)$.
3. D is called local if $(Dw)|_U$ depends only on $w|_U \; \forall U \subset M$ open.

Lemma:

1. Let D_j be derivations, A_j Anti-derivations ($j = 1, 2$). Then
   - $[D_1, D_2] = D_1 D_2 - D_2 D_1$ (Kommutator)
   - $\{A_1, A_2\} = A_1 A_2 + A_2 A_1$ (Anti-Kommutator)

   are again derivations while $[D_j, A_k] = D_j A_k - A_k D_j$ is an anti-derivation.

2. Two derivations or anti-derivations are equal if they coincide already on 0-forms and 1-forms

3. If D is a local derivation or anti-derivation that commutes with d then D is already determined by its action on 0-forms.

Proof:

1. For example if A_j has a degree d_j and $w \in \Lambda_k(M), \sigma \in \Lambda_l(M)$ then

$$\begin{aligned}\{A_1, A_2\}(w \wedge \sigma) &= A_1\left((A_2 w) \wedge \sigma + (-1)^{d_2 \cdot k} w \wedge (A_2\sigma)\right) + (\text{ same but } 1 \leftrightarrow 2) \\ &= (A_1 A_2 w) \wedge \sigma + (-1)^{d_1(k+d_2)}(A_2 w) \wedge (A_1\sigma) + (-1)^{d_2 \cdot k}(A_1 w) \wedge (A_2\sigma) + \\ &\qquad (-1)^{k(d_1+d_2)} w \wedge (A_1 A_2 \sigma) + (\text{ same but } 1 \leftrightarrow 2) \\ &= [(A_1A_2 + A_2A_1)w] \wedge \sigma + (-1)^{k(d_1+d_2)} w \wedge [(A_1A_2 + A_2A_1)\sigma] + \\ &\qquad (A_2 w) \wedge (A_2\sigma)\left[(-1)^{d_1(kd_2)} + (-1)^{d_1 k}\right] + \\ &\qquad (A_1 w) \wedge (A_2\sigma)\left[(-1)^{d_2 k} + (-1)^{d_2(k+d_1)}\right] \\ &= (\{A_1, A_2\} w) \wedge \sigma + (-1)^{k(d_1+d_2)} w \wedge (\{A_1, A_2\}\sigma) \\ &= (\{A_1, A_2\} w) \wedge + w \wedge (\{A_1, A_2\}\sigma)\end{aligned}$$

which we wanted to show, The calculations for $[D_1, D_2], [D_j, A_k]$ are similar

2. We use again that w is a linear combination of the forms $f_0 w_1 \wedge \cdots \wedge w_n$. Now the statement follows from the Leibniz or Anti-Leibniz rule.

3. If also $[D, d]$ vanishes, then apply this to w of the form $f_0 \,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ $\rightarrow D$ already determined by its action on 0-forms.

Theorem:

On $\Lambda(M)$ we have the following identities between $\mathrm{d}, i_v, \mathcal{L}_v$

1. $\mathcal{L}_v = i_v \,\mathrm{d} + \mathrm{d} i_v$
2. $[\mathcal{L}_u, i_v] = i_{[u,v]}$
3. $[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]}$

Proof:

Observations:

- i_v: Antiderivation of degree -1
- d: Antiderivation of degree +1
- $\mathcal{L}_v$ Derivation of degree 0

1. $\mathrm{d}\mathcal{L}_v w = \mathrm{d}\left[\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^v)^* w\right]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{d}(\psi_t^v)^* w\right]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[(\psi_t^v)^*\,\mathrm{d}w\right] = \mathcal{L}_v\,\mathrm{d}w$
   $\Rightarrow [\mathcal{L}_v, \mathrm{d}] = 0$ thus we are in case 3) of the 2. Lemma, so it remains to check $\mathcal{L}_v = i_v\,\mathrm{d} + \mathrm{d}i_v$ on functions:
   $$(i_v\,\mathrm{d} + \mathrm{d}i_v)f = i_v\,\mathrm{d}f = v\,[f] = \frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^v)^* f = \mathcal{L}_v f$$

2. By using the 2nd Lemma we know that $[\mathcal{L}_v, i_v]$ is an anti-derivation in its 0-form:
   $$[\mathcal{L}_u, i_v]\,f = -i_v(\mathcal{L}_u f) = 0 = i_{[u,v]} f = 0$$
   1-forms: (linear combinations of the form $g\,\mathrm{d}f\ ,\ f, g \in C^\infty(M)$
   $$\begin{aligned}[\mathcal{L}_v, i_v]\,g\,\mathrm{d}f &= \mathcal{L}_u(i_v(g\,\mathrm{d}f)) - i_v(\mathcal{L}_u(g\,\mathrm{d}f))\\ &= \mathcal{L}_u(gv\,[f]) - i_v((\mathcal{L}_u g)\,\mathrm{d}f + g(\mathcal{L}_u\,\mathrm{d}f))\\ &= u\,[gv\,[f]] - ((\mathcal{L}_u g)v\,[f] - g i_v(\mathcal{L}_u\,\mathrm{d}f))\\ &= u\,[g]\,v\,[f] + gu\,[v\,[f]] - u[g]v[f] - g i_v(i_u\,\mathrm{d} + \mathrm{d}i_u)\,\mathrm{d}f\\ &= g(u\,[v[f]]\,(i_v\,\mathrm{d}i_u\,\mathrm{d}))\\ &= g([u,v])[f] = g i_{[u,v]}\,\mathrm{d}f = i_{[u,v]}(g\,\mathrm{d}f)\end{aligned}$$

3. Since $[\mathrm{d}, \mathcal{L}_v] = 0$ it is sufficient to check on 0-forms:
   $$[\mathcal{L}_u, \mathcal{L}_v]\,f = (\mathcal{L}_u\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_u)(f) = u\,[v[f]] - v\,[u[f]] = \mathcal{L}_{[u,v]} f$$

Application:
Classical mechanics on $\mathbb{R}^{2m}$ is formulated in terms of Poisson brackets:
$$\{f, g\} = \frac{\partial f}{\partial p_a}\frac{\partial f}{\partial q^a} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a}$$
with generalized coordinates q^a and p_a.
$X\ f \cdot g = \{f, g\}$ is called the Hamiltonian vector field of $f \in C^\infty(\mathbb{R}^{2n})$
For $f = H$ Hamiltonian of the system the equations of motion $\dot{q}^a = \{H, q^a\}\ ,\ \dot{p}_a = \{H, p_a\}$ are nothing but the integral curves of X_H. All of this can be generalized to any even dimensional manifold M as follows. On M we require a non-degenerate 2-form w which is closed. $\mathrm{d}w = 0$, non-degenerate means that $i_u w = 0$, implies $u = 0$ for any vector field u on M. Given $f \subset C^\infty(M)$ we define the Hamiltonian vector field X_f of f by the formula $i_{X_f} w + \mathrm{d}f = 0$ given X_f because w non-deg.. $\{f, g\} = X_f[g]$ action of vector fields. $\mathrm{d}w = 0$ closed $\Leftrightarrow$ Jacobi identity $\{f, \{g, h\}\} +$ cyclic $= 0$. Recall that a map $\psi : M \to M$ is called a canonical transformation provided by it preserves Poisson brackets.
$$\begin{aligned}\{f(\psi(m)), g(\psi(m))\} &= \{f, g\}\,(\psi(m))\forall m \in M\ ,\ f, g \in C^\infty(M)\\ &\Leftrightarrow \{\psi^* f, \psi^* g\} = \psi^*\{f, g\}\\ &\Leftrightarrow \psi^* w = w \text{ (symplectomorphism)}\end{aligned}$$

if the canonical 2-form is preserved.
Let now a 1-parameter family ψ_s^u of diffeos be given which are also symplectomorphisms.

$$\Rightarrow (\psi_s^u)^* w = w$$

$$\mathcal{L}_u w = 0 = i_u \, \mathrm{d}w + \mathrm{d}i_u w = \mathrm{d}i_u w = 0$$

$\Rightarrow$ locally $\exists$ 0-form f such that $i_v w + \mathrm{d}f = 0$. In other words $v = X_f$ Hamiltonian vector fields are generations of canonical transformations.

7 Stoke's Theorem, Poincarè Lemma, de Rham Cohomology

Objective: Develop integral calculus for $\Lambda_n(M)$. M supposed to be paracompact. There exist a partition of unity $\{e_I\}_{I\in\mathcal{I}}$ subordinate to a choice of atlas (U_I, x_I) of M. i.e. $e_I \in C^\infty(M) \ni$

1. $0 \leq e_I \leq 1$
2. $\overline{\mathrm{supp}(e_I)} \subset U_I$ $(\mathrm{supp}(f) = \{p \in M, f(p) \neq 0\})$
3. $(\sum_I e_I)(p) = 1 \forall p \in M$

Suppose that an m-form on M , $\dim(M) = m$ has compact support, i.e. $\overline{\mathrm{supp} w}$ compact. Then we want to define for $w \in \Lambda_m(M)$

$$\int_M w := \int_M (\sum_I w_I) w = \sum_I \int_M e_I w$$

(U_I) is a cover of $\mathrm{supp}(w)$ which is compact (or contained therein) so $w e_I \neq 0$ for finitely many $I \in \mathcal{I}$.

$$\int_M w = \sum_I \int_M e_I w = \sum_I \int_{U_I} e_I w \text{ as } \overline{\mathrm{supp}(e_I)} \subset U_I$$

As $\int_{U_I} e_I w$ is now defined over a single chart U_I we can use coord. methods to define the integral

$$\int_{U_I} e_I w = \int_{X_I(U_I)} \frac{1}{m!} (e_I)_I(x) w_{I_{\mu_1,\dots,\mu_m}}(x) \, \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_m}$$
$$= \int_{x_I(U_I)} (e_I)_I(x) w^{(x)}_{I_{1,\dots,m}} \, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m$$
$$:= \int_{X_I(U_I)} (e_I)_I(x) w_{I_{1,\dots m}}(x) \, \mathrm{d}^m x$$

integral over a subset of $\mathbb{R}^m$. Is it well defined?

1. Independence of choice of atlas
2. Independence of choice of partition of unity

If M paracompact with partition of unity $(e_I)_{I\in\mathcal{I}}$ subordinate to (U_I, x_I) then

$$\int_M \omega := \sum_I \int_{x_I(U_I)} (e_I)_I(x)\omega_{I_{1-M}} \underbrace{\mathrm{d}^M x}_{Lebesquemeasure}$$

where $(e_I)_I, \omega_{I_{\mu_1,\ldots,\mu_m}}$ are the local coordinate expressions for the e_I with compact support in U_I and ω on U_I.
To check independence of the partition unity and choice of atlas:

Proof:

Let $(V_J, y_J)_{J\in\mathcal{J}}$ be a equivalent altas with partition of unity f_J subordinate to it. Then we must establish

$$\sum_I \int_{x_I(U_I)} (e_I)_{\mathcal{I}}(x)\omega_{I_{1,\ldots,m}}(x)\,\mathrm{d}^m x = \sum_J \int_{y_J(V_J)} (f_J)(y)\omega_{J_{1,\ldots,m}}(y)\,\mathrm{d}^m y$$

To see this note that from $1 - \sum_I e_I(p) \forall p \in M$ we have in particular

$$1 = \sum_I (e_I)_J(y) \forall y \in y_J(V_J); (e_I)_J = e_I \circ y_J^{-1}$$

likewise

$$1 = \sum_J (f_J)_I(x) \forall x \in x_I(U_I)$$

e_I is defined on $y_J(V_J)$ and since e_I by definition has compact support in U_I, $(e_I)_J$ has compact support in $y_J(U_I) \cap y_I(V_J) = y_J(U_I \cap V_J)$. By inserting the function $\equiv 1$ on both sides of the „=“ we find

$$\sum_{I,J} \int_{x_I(U_I\cap V_J)} \mathrm{d}^m x (e_I)_{\mathcal{I}}(x)(f_J)_I(x)\omega_{I_{1,\ldots,m}}(x) \stackrel{!}{=} \sum_{I,J} \int_{y_J(U_I\cap V_J)} \mathrm{d}^m y (f_J)(y)(e_I)_J(y)\omega_{J_{1,\ldots m}}$$

The logic behind this was to write both sides as integrals over images of the same sets $U_I \cap V_J$. Note that if $U_I \cap V_J = \varnothing$ then the integrands vanish anyway, consider the diffeomorphism $\varphi_{IJ} = y_J \circ x_I^{-1}$, $x_I(U_I, V_J) \to y_J(U_I \cap V_J)$. By the transformation law of the Lebesque measure under changes of coordinates we have

$$\mathrm{d}^m y = \left| \underbrace{\det\left(\frac{\partial\varphi(x)}{\partial x}\right)}_{\text{Jacobean always positive}} \right| \mathrm{d}^m x$$

Also by the transformation law of functions and m-forms:

$$(f_J)_J(y)|_{y=\varphi_{IJ}(x)} = \left(f_J \circ \varphi_J^{-1}\right)(\varphi_{IJ}(x)) = \left(f_J \circ y_J^{-1}\right)\left(y_J \circ x_I^{-1}(x)\right) = \left(f_J \circ x_I^{-1}\right)(x) = (f_J)_I(x)$$

Likewise

$$(e_I)_J(x)|_{y=\varphi_{IJ}(x)}\,(e_I)_J(x)$$

$$\Rightarrow \sum_{I,J}\int_{x_I(U_I\cap V_J)} \mathrm{d}^m x (e_I)_{\mathcal{I}}(x)(f_J)_I(x)\omega_{I_{1,\ldots,m}}(x) \overset{!}{=}$$

$$\sum_{I,J}\int_{x_I(U_I\cap V_J)} \mathrm{d}^m x (e_I)_{\mathcal{I}}(x)(f_J)_I(x)\left|\det\left(\frac{\partial\varphi_{IJ}(x)}{\partial x}\right)\right|\omega_{J_{1,\ldots,m}}(\varphi_{IJ}(x))$$

By definition for $p \in U_I \cap V_J$ we have m:

$$\omega(p) = \omega_{J_{\mu_1,\ldots,\mu_m}}(y)\,\mathrm{d}y^{\mu_1}\wedge\cdots\wedge\mathrm{d}y^{\mu_m}$$

$$= \omega_{J_{\mu_1,\ldots,\mu_m}}(\varphi_{IJ}(x))\,\frac{\partial\varphi_{IJ}^{\mu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial\varphi_{IJ}^{\mu_m}}{\partial x^{\mu_m}}\,\mathrm{d}x^{\nu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_m}$$

$$\omega_{I_{\mu_1,\ldots,m}}(x)\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_m}$$

$$\omega_{I_{1,\ldots,m}}(x) = \omega_{J_{\mu_1,\ldots,\mu_m}}(\varphi_{IJ}(x))\,\frac{\varphi_{IJ}^{\mu_1}}{\partial x^{\mu_1}}\cdots\frac{\partial\varphi_{IJ}^{\mu_m}}{\partial x^{\mu_m}}$$

$$= \omega_{J_{\mu_1,\ldots,\mu_m}}(\varphi_{IJ}(x))\,\frac{\partial\varphi^{\mu_1}}{\partial x^1}\cdots\frac{\partial\varphi^{\mu_m}}{\partial x^m}$$

$$= \omega_{J_{1,\ldots,m}}(\varphi_{IJ}(x))\,\epsilon_{\mu_1\ldots\mu_m}\frac{\varphi^{\mu_1}}{\partial x^1}\cdots\frac{\partial\varphi^{\mu_m}}{\partial x^m}$$

$$= \omega_{J_{1,\ldots,m}}(\varphi_{IJ}(x))\det\left(\frac{\partial\varphi_{IJ}(x)}{\partial x}\right)$$

So both expressions are equal if $\det\left(\frac{\partial\varphi_{IJ}(x)}{\partial x}\right) > 0 \forall U_I \cap V_J \neq \varnothing$. The manifold we are integrating over must be in particular oriented. Change $V_I = U_I, x_I = y_I$ then φ_{IJ} are just passive diffeos, thus M is orientated by defenition.
$\Rightarrow$ The integral of an m-form over an m-dim. manifold is only well-defined if M is orientable.

Remark:

1. It is not difficult to show that an m-dim. manifold is orientable $\Leftrightarrow$ M admits a nowhere vanishing m-form.
   Application: Any symplectic manifold is in particular orientable. Recall that a symplectic manifold M is equipped with a closed and non-degenerate 2-form ω. Thus $m = \dim(M)$ is even. Thus M comes equipped with the nowhere vanishing Liouville m-form
   $$\Omega = \underbrace{\omega\wedge\cdots\wedge\omega}_{\text{factors}}$$
2. Another application of above proof of independece of choice of altas is the case of an active diffeo $\psi : M \to M$ with altas $(V_I = \psi^{-1}(U_I), y_I = x_I \circ \psi)$
   $$\int_M \omega = \int_{\psi(M)} \omega = \int_M (\psi^*\omega)$$
   by the very same apllication.

Theorem:

Stokes:

Let $\omega \in \Lambda_{m-1}(M), \dim(M) = m$

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

$$\int_{[a,b]} \mathrm{d}f = \int_a^b \mathrm{d}x \frac{\partial f}{\partial x} = f(b) - f(a) = \int_{\partial[a,b]} f$$

Proof:

By using the definition of a manifold M with boundry ∂M and a partition of unity for M:

$$\int_{\partial M} \omega = \sum_I \int_{\partial M} e_I \omega = \sum_I \int_{U_I \cap \partial M} e_I \omega$$

Two cases:

1. $\partial M \cap U_I = \varnothing$. Then both $\int_{\partial M \cap U_I} e_I \omega = 0$ and $\int_{U_I} \mathrm{d}(e_I \omega) = 0$ because

$$\begin{aligned}\int_{U_I} \mathrm{d}(e_I \omega) &= \int_{x_I(U_I)} \frac{1}{(m-1)!} \partial_{\mu_1} \left((e_I)_I \omega_{I_{\mu_2,\dots,\mu_m}} \right) \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_m} \\ &= \int_{x_I(U_I)} \frac{1}{(m-1)!} \epsilon^{\mu_1,\dots,\mu_m} \left((e_I)_I \omega_{I_{\mu_2,\dots,\mu_m}} \right) \mathrm{d}^m x \\ &= \sum_{\mu_1=1}^{m} \frac{1}{(m-1)!} \int_{x_I(U_I)} \mathrm{d}^m x \frac{\partial}{\partial x^{\mu_1}} \underbrace{\left[\epsilon^{\mu_1,\dots,\mu_m} (e_I)_I \omega_{I_{\mu_2,\dots,\mu_m}} \right]}_{f_I^{\mu_1}}\end{aligned}$$

   Apply fundamental theorem of integral calculus, then we get m integrals over the boundry of $x_I(U_I)$, but on $\partial x_I(U_I) : f_I^{\mu_1} \equiv 0 \forall \mu_1 = 1, \dots, m$ because e_I has compact support in the interior of U_I so cannot touch ∂U_I

$$\Rightarrow \int_{U_I} \mathrm{d}(e_I \omega) = 0$$

2. $\partial M \cap U_I \neq \varnothing$: By assumption

$$x_I(U_I \cap \partial M) \subset \partial \mathbb{R}^m_- = \left\{ x \in \mathbb{R}^m, x^1 = 0 \right\}$$
$$x_I(U_I \cap M) \subset \mathbb{R}^m_- = \left\{ x \in \mathbb{R}^m, x^1 \leq 0 \right\}$$

$$\Rightarrow \int_{U_I} \mathrm{d}(\omega e_I) = \frac{1}{(m-1)!} \sum_{\mu_1=1}^{m} \int_{x_I(U_I)} \mathrm{d}^m x \epsilon^{\mu_1,\dots,\mu_m} \partial_{\mu_1} (e_I)_I \omega_{I_{\mu_2,\dots,\mu_m}}$$

$$= \frac{1}{(m-1)!} \int_{x_I(U_I \cap \partial M)} \mathrm{d}^{m-1} x (e_I)_I (x^1 = 0, x^2, \dots, x^m) \omega_{I_{\mu_2,\dots,\mu_m}} (0, x^2, \dots, x^m) \epsilon^{1,\mu_2,\dots,\mu_m}$$

$$= \frac{1}{(m-1)!} (m-1)! \int_{x_I(U_I \cap \partial M)} \mathrm{d}^{m-1} x e_I (x^1 = 0, x^2, \dots, x^m) \omega_{I_{2,\dots,m}} (x^1 = 0, x_2, \dots, x^m)$$

$$= \int_{U_I \cap \partial M} e_I \omega$$

since ∂M is equipped with the atlas $V_I = \partial M \cap U_I$ and $y^\mu_I = x^\mu_I$ for $\mu \neq 2, \dots, m$.

Thus in both cases $\int_{U_I \cap \partial M} e_I \omega = \int_{U_I} \mathrm{d}(e_I \omega)$. Summing both sides over I:

$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega$$

Remark:

1. Let $\psi : N \to M$ be an embedding so $\dim(N) = n$ do not need to be identical. Let $w \in \Lambda_{n-1}(M)$ then $\psi^* \in \Lambda_{n-1}(N)$

$$\int_{\partial N} \psi^* \omega = \int_N \mathrm{d}\psi^* \omega = \int_N \psi^* \,\mathrm{d}\omega = \int_{\psi(\partial N)} \omega = \int_{\psi(N)} \mathrm{d}\omega$$

   This defines the integrals over embedded submanifolds $N \cap M$

2. So far we assumed ω to have compact support. It is sufficient that ω has sufficient drop-off behaviour if M is not compact itself.

3. CAREFUL: It is an innocent looking assumption in Stoke's theorem that ω is smooth on M. If one violates, one gets a contradiction.

   Example:

   (Counter-example:)

$$\omega = \frac{\epsilon_{\mu\nu} x^\nu \,\mathrm{d}x^\mu}{(x^1)^2 + (x^2)^2}$$

   Smooth everywhere except at the origin $M = D \subset \mathbb{R}^2$
   (D is unit-circle: $\partial D = \{x \in \mathbb{R}^2, (x^1)^2 + (x^2)^2 = 1\}$)

$$\mathrm{d}\omega\, \mathrm{d}\left(\frac{x^1 \,\mathrm{d}x^2 - x^2 \,\mathrm{d}x^1}{\underbrace{(x^1)^2 + (x^2)^2}_{r^2}} \right) = \left(\frac{\partial}{\partial x^1} \frac{x^1}{r^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \left(\frac{\partial}{\partial x^2} \frac{x^2}{r^2} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2$$
$$= \left(\frac{1}{r^2} + \frac{1}{r^2} - \frac{2}{r^4}(x^1)^2 - \frac{2(x^2)^2}{r^4} \right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 = 0$$

   It is closed everywhere except at $r = 0$ where it is ill defined. This means that Stoke's theorem is not applicable but suppose forget about it and "smoothing extendet $\mathrm{d}\omega$ also to $r = 0$":

$$0 = \int_D \mathrm{d}\omega \int_{\partial D} \omega = \int_0^{2\pi} \frac{\cos(t)\,\mathrm{d}(\sin(t)) - \sin(t)\,\mathrm{d}(\cos(t))}{(\cos(t))^2 + (\sin(t))^2} = \int_0^{2\pi} \mathrm{d}t = 2\pi$$

At ∂D we have $x^1 = \cos(t), x^2 = \sin(t), t \in [0, 2\pi)$. $\mathrm{d}\omega$ cannot be extended as a smooth function to D but as a distribution

$$\mathrm{d}\omega = 2\pi\delta(x)\,\mathrm{d}x^1\,\mathrm{d}x^2$$

where $\delta(x)$ is the δ-distribution defined as a linear function on a space of smooth functions by

$$\delta[f] \coloneqq \mathrm{d}^2x\delta(x)f(x) = f(0)$$

4. This is the starting of algebraic gemoetry and topology, for instance:

   Example:

   Let ω be a p-form, closed (cocycle) if $\mathrm{d}w = 0$, exact (coboundary) if $\omega = \mathrm{d}\sigma$. Let N be a p-dim. sub-manifold: cycle: $\partial N = \varnothing$, boundary: $N = \partial Q$

   - $B_p(M) \subset Z_p(M) \subset \Lambda_p(M)$ (exact $\subset$ closed $\subset$ all forms)
   - $B^p(M) \subset Z^p(M) \subset \Lambda^p(M)$ (boundaries $\subset$ cycles $\subset$ all p submanifolds)
   - $H^p(M) = Z^p(M)/B^p(M)$ "Homology Groups"
   - $H_p(M) = Z^p(M)/b_b(M)$

   Period integrals $\langle N, w\rangle \coloneqq \int_N \omega = \langle [N], [\omega]\rangle$ independent of the representative

Theorem:

(de Rham):
$\dim(H_p(M)) = \dim(H^p(M)) =: b_p(M)$: p-th Beti number of M (M compact) related to topological properties of M.

Period integral $\langle N, \omega\rangle = \int_N \omega, \omega \in \Lambda_n(M), \dim(N) = n$, N oriented submanifold of M. Formal addition of submanifolds with integer coefficients gives rise to the Abelan group $\Lambda^n(M)$. A typical d of $\Lambda_n(N)$ is given by $N = \sum_{i=1}^{L} z_i N_i$ where $N_i \cap N_j = \varnothing$. $z_i \in \mathbb{Z}$ called coliding number .

$$\langle N, \omega\rangle = \sum_i z_i \langle N, \omega\rangle$$

We can form subgroups consisting of elements with $\partial N = \varnothing$ or $N = \partial Q$

$$H^n = Z^n(M)/B^n(M)$$
$$[N] = \left[N + N'; N' = \partial Q, Q \in \Lambda^{n+1}(M)\right]$$
$$[\omega] = \{\omega + \mathrm{d}\sigma, \sigma \in \Lambda_{n-1}(M)\}$$
$$\langle [N], [\omega]\rangle \coloneqq \int_N \omega$$
$$b_p(M) \coloneqq \mathrm{dim} H_p(M) = \dim(H^p(M))$$

Let S be a simplicat decomposition of M (p-simplex is the higher dimensional analog if a triangled tetrahedron in $\frac{2}{3}$ dimensions)
Let $N_p(S) \coloneqq \#$ of p-dim simplices appearing in S. Then:

$$\chi(M) \coloneqq \sum_{p=0}^{\dim(M)} (-1)^p N_p(S)$$

is called the Euler characteristic of M and independent of the choice of S.

Theorem:

$$\chi(M) = \sum_{p=0}^{\dim(M)} (-1)^p b_p(M)$$

Thus there is a deep interplay between “global“ features of M and infinitessimal concepts such as differential forms.

Lemma:
(Poincarè)
Suppose that $U \subset M$ is contractible to a point $p_0 \in M$, i.e., $\exists$ smooth map $F[0,1]xU \to U$ with $F(0,p) = p_0, F(1,p) = p$. Then any closed form U is also exact.

Proof:
In suitible coordinates we can assume $x(p_0) = 0$ and that U lies in a single chart (otherwise subdivide). We want do define $\sigma \ni \omega = \mathrm{d}\omega$. A possible choice is

$$\sigma = \int_0^1 \mathrm{d}t \frac{t^{n-1}}{(n-1)!} x^\nu \omega_{\nu\mu_1,\ldots,\mu_{n-1}}(t,x)\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{n-1}}$$

Check $\mathrm{d}\sigma = \omega$ as an exercise, use $\mathrm{d}\omega = 0$.
This means that de Rham cohomology is a global concept, locally it is trivial.

8 Riemannian Geometry

Motivation for the concept of the covariant differential.

Example:
$M = S^2$ is a 2-dimensional manifold,

$$M = \left\{x \in \mathbb{R}^3; ||x||^2 = 1\right\}, x^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$$

Local coordinates on S^2 are the spherical coordinates $\{y^1, y^2\} = \{\theta, \varphi\}$

$$\psi : S^2 \to \mathbb{R}^3 : (\theta, \varphi) \mapsto (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$$

Tangent vectors to M are given by $b_\mu = \frac{\partial \psi}{\partial y^\mu}, \mu = 1, 2$ since $\vec{\psi}\frac{\partial \vec{\psi}}{\partial y^\mu} = \frac{1}{2}\frac{\partial}{y^\mu}(\vec{\psi}^2) = 0$
b_μ varies from point to point on S^2 therefore this basis of tangent vectors will change when we move on the sphere.
We can in fact explicitly compute the way b_μ changes as we vary $y \in S^2$ by computing the corresponding derivative

$$\frac{\partial}{\partial y^\mu} b_\mu \text{ we find}$$

$$\frac{\partial}{\partial y^1} b_1 = -\psi, \frac{\partial}{\partial y^2} b_1 = \cot(\theta) b_2$$

$$\frac{\partial}{\partial y^1} b_2 = \cot(\theta) b_2, \frac{\partial}{\partial y^2} b_2 = -\sin(\theta)\cos(\theta) b_1 - \sin^2(\theta)\psi$$

b_1, b_2, ψ forms a basis of $\mathbb{R}^2$.
Now a function on S^2 can be considered as a function on $\mathbb{R}^3$ which does not depend on the radial coordinate, or in other words which satisfies $\psi^a \frac{\partial f}{\partial x^a} = 0$

$$\psi^a \frac{\partial f}{\partial x^a} = 0 = \frac{x^a}{r}\frac{\partial f}{\partial x^a} = \frac{\partial f}{\partial r}$$

We consider fields $\frac{\partial}{\partial y^\mu}$ as the derivations $\frac{\partial \psi^a}{\partial y^\mu}\frac{\partial f}{\partial x^a}$ on functions satisfying $\psi^a \frac{\partial f}{\partial x^a} = 0$ and compute the change of this derivation

$$\frac{\partial}{\partial y^\mu}\partial_\nu = \frac{\partial b^a_\nu}{\partial y^\mu}\frac{\partial}{\partial x^a} = \Gamma^\rho_{\mu\nu}\partial_\rho + \mathcal{H}_{\mu\nu}\psi^a\frac{\partial}{\partial x^a}$$

Applied to functions on the sphere we obtain

$$\left(\frac{\partial}{\partial y^\mu}\partial_\nu\right) f = \Gamma^3_{\mu\nu}\partial_\rho$$

$$\Gamma^\rho_{11} = 0\ ,\ \Gamma^\rho_{21} = \cot(\theta)\delta^\rho_2 \nabla = \Gamma^\rho_{12}\ ,\ \Gamma^\rho_{22} = -\sin(\theta)\cos(\theta)\delta^\rho_1$$

So the coordinate vectorfields ∂_μ defined as above change in direction μ by the amount $\Gamma^\rho_{\mu\nu}\partial_\rho$.

For more general vector fields we have

$$\partial_\mu(v^\nu\partial_\nu) = \frac{\partial v^\nu}{\partial y^\mu} + v^\rho\Gamma^\nu_{\mu\rho}\partial_\nu = \left(\partial_\mu v^\nu + \Gamma^\nu_{\mu\rho}v^\rho\right)\partial_\nu := (\nabla_\mu v)^\nu\partial_\nu$$

$(\nabla_\mu v)^\nu$ is the so called covariant derivative of v in direction μ where the second term is due to the explicit dependence of the coordinate fields on the sphere. Since the object $v^\mu\partial_\mu$ is globally defined, also $\partial_\mu v$ must be globally defined and the components $(\nabla_\mu v)^\nu$ define a new tensor which would not be the case if we would just consider the first term $\frac{\partial v^\nu}{\partial y^\mu}$.