Particle- and Astroparticlephysics

Advanced Experimental Physics

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Contents

1 Recap of particle physics

1.1 Units

Natural units:

- $$kg, m, s$$

  replaced by units of GeV

- $\hbar = c = 1$

Consequences:

1. $E^2 = \vec{p}^2 c^2 + m^2 c^4 \Rightarrow E^2 = \vec{p}^2 + m^2$
2. Energy, momentum, mass: $[E]^1$
3. Length, time: $[E]^{-1}$, e.g., $1\,\mathrm{cm} = \frac{1\,\mathrm{cm}}{\hbar c} = 51\,000\,\mathrm{eV}$
4. Crosssection: $[E]^{-2}$

Example 1.1.

$$\hbar c \simeq 200\,\mathrm{MeV}\cdot\mathrm{fm} = 1 \quad 1\,\mathrm{fm} = \frac{5}{\mathrm{GeV}}\hbar \simeq 6.6\times 10^{-25}\,\mathrm{GeV}\cdot\mathrm{s} = 1$$

1.2 Realtivistic Kinematics

Most important transformation is the Lorentz transformation (relates spacetime coordinates in different inertial systems):

$$A^\mu := \left(A^0, \vec{A}\right)$$

contravariant four vectors in Minkowski metric.

$$A_\mu := \left(A^0, -\vec{A}\right)$$

covariant four vector. $\mu = 0, 1, 2, 3$

Review of four vectors

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = x^{\mu'} = \Lambda^\mu_\nu x^\nu$$

(Remember: sum convention, sum over over identical upper and lower indices)

Lorentz invariant space-time interval $t^2 - x^2 - y^2 - z^2$ can be expressed as a four vector scalar product by defining the covariant space-time four vector

$$x_\mu = (t, -x, -y, -z)$$

Scalar product

$$x^\mu x_\mu = x^0 x_0 + x^1 x_1 + x^2 x_2 + x^3 x_3 = t^2 - x^2 - y^2 - z^2$$

$\Rightarrow$ Scalar products are lorentz-invariant. Can write:

$$\begin{pmatrix} t' \\ -x' \\ -y' \\ -z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & +\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ +\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} t \\ -x \\ -y \\ -z \end{pmatrix} \Rightarrow x'_\mu = \Lambda^\nu_\mu x_\nu$$

$$x_\mu = g_{\mu\nu} x^\nu$$

with

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

the metric tensor of the Minkowski metric.
Scalar product: $A \cdot B = A^\mu B_\nu = A_\mu B^\mu = g_{\mu\nu} A^\mu B^\nu = A^0 B^0 - \vec{A}\vec{B}$ is a basic invariant under Lorentz transformation.

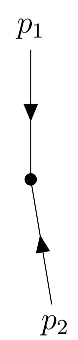
Example 1.2. Scalar product of four-momentum

$$p^\mu = (E, \vec{p})$$

$$p^2 = p^\mu p_\mu = \sum_{\mu=0}^{3} p^\mu p_\mu = E^2 - p^2$$

is lorentz-invariant because of energy-momentum relation $E^2 = \vec{p}^2 + m^{\Rightarrow} p^2 = m^2$. $\Rightarrow$ Rest mass is the same in all reference frames.

Example 1.3. Total energy $\sqrt{s}$ of a particle collision



the quantity

$$s := (p_1 + p_2)^\mu \, (p_1 + p_2)_\mu$$

is lorentz-invariant (the same in all reference frames).

Four-vector derivatives

$$\partial_\mu = \frac{\partial}{\partial x^\mu} := \left(\frac{\partial}{\partial t}, \vec{\nabla} \right)$$

transforms a covariant vector (uses Λ_μ^ν).

$$\partial^\mu := \frac{\partial}{\partial x_\mu} := \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

transforms as a contravariant vector. Scalar product:

$$\Box := \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$$

d'Alembert operator.

1.3 Elementary particles

Fermions (spin $\frac{1}{2}$).

Lepton sector

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \nu^- \end{pmatrix}$$

Upper particles: $q = 0$, lower $q = -e$. Know today that $m(\nu) > 0$ (but very small) $\rightarrow$ neutrino oscillations.

Quark sector

$$\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$$

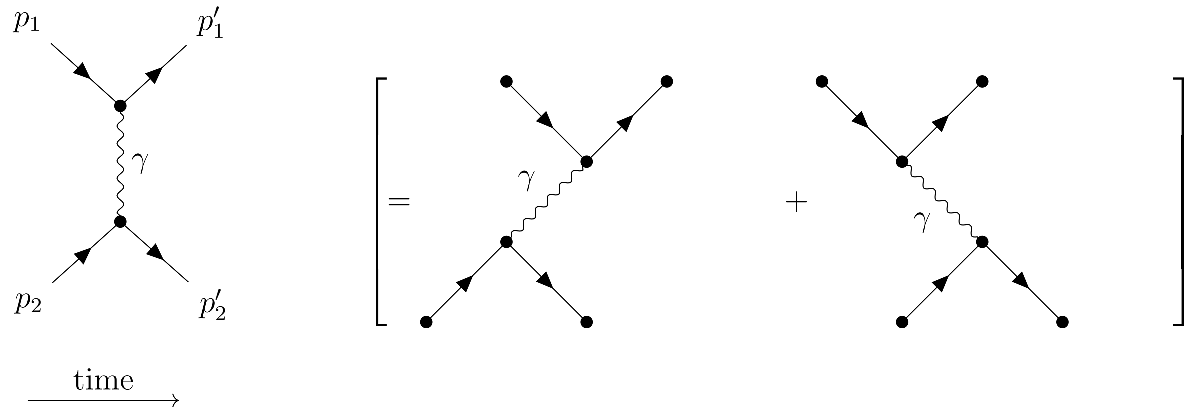
Upper particles: $q = \frac{2}{3}e$, lower particles: $q = -\frac{1}{3}$ free particles are bound states (hadrons) either baryons (qqq), mesons ($q\bar{q}$), color neutral. Result of so called confinement.
Additionally:

- 6 anti-leptons
- 6 anti-quarks

Bosons Exchange particles, spin 1.

- γ: e.m. interaction, $m = 0$, $q = 0$, couples to e.m. charge, no self-coupling
- Z^0: Weak interaction, $m = 91\,\text{GeV}$, $q = 0$
- W^+W^- couples to weak charge, $m = 80\,\text{GeV}$, $q = \pm 1$
- g: Strong interaction, $m = 0$, $q = 0$ (8 in total) couples to color charge, carry color charge $\rightarrow$ self-coupling
- Higgs-boson H^0: gives mass to $Z^0, W^\pm$, $m = 125\,\text{GeV}$ $\rightarrow$ Higgs-mechanism

Example 1.4. E.m. scattering, leading order



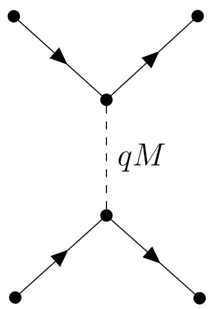
four momentum transfer: $q^2 := (p_1 - p_1')^2$

- Difference between the four-momenta entering and leaving the interaction region
- four-momentum of virtual particle

Because the q^2-dependence is determined by 4-momenta of incoming and outgoing particles virtual particle does not obey energy-momentum relation:

$$q^2 \neq m_x^2$$

Quantum mechanical transition amplitude for this process to happen:

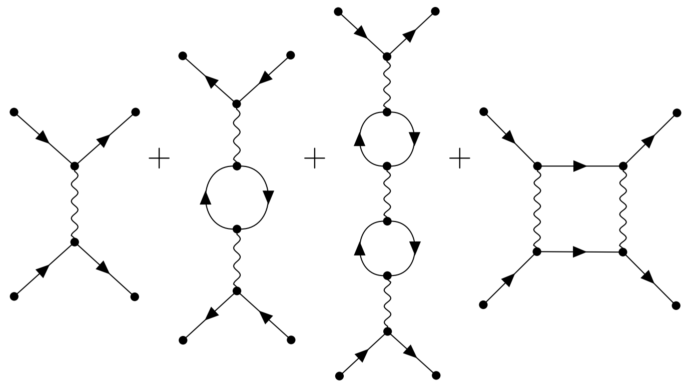


amplitude $\sim g_1 \frac{1}{q^2 - M^2} g_2$. g_i: Coupling strength between particle and exchange boson, fraction: boson propagator for exchange particle with mass M.

Fermi's Golden Rule
Cross-section $\sim |$ amplitude $|^2$.

$$\sim \frac{g_1^2 g_2^2}{q^4}$$

for high momentum transfer ($|q|^2 >> M^2$). Full scattering amplitude? E.m. interaction:



$$\sim l^2 \qquad \sim l^4 \qquad \sim l^6 \cdots$$

Natural units

Definition 1.1. Finestrucuter constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$$

2 Covariant description of relativistic particles

Definition 2.1. Interaction: Interplay of wavefunction of one particle (probe) in the potential of another particle (target).

Test the particle (and underlying theory) by measuring cross sections, angular distribution, branching ratios and decay rates. Need:

- proper equation of motion
- calculation of transition amplitude

Will do that for QED as example. Need for computation:

- relativistic description
- accounting for particle spin

2.1 Non-relativistic Quantum Mechanics

Energy-momentum relation:

$$E = \frac{|\vec{p}|^2}{2m}$$

identify

$$E \to i\hbar \frac{\partial}{\partial t}$$
$$\vec{p} \to -i\hbar \vec{\nabla}$$

Operator equation:

$$\left(i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \phi\left(\vec{x}, t\right) \right) = 0$$

Schrödinger equation of free particle.

Statistical interpretation: $|\phi|^2 \, \mathrm{d}^3x$ is probability to find particle in volume d^3x around $\vec{x}$ at time t.

$$\rho = \rho(\vec{x}, t) \;\; : \;\; |\phi(\vec{x}, t)|^2$$

is a localisation probability density.

Definition 2.2. Continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0$$

$\vec{j}$ is the probability density current.

How does $\vec{j}$ depend on ϕ?

$$-i\phi^* \cdot \left(i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \phi\left(\vec{x}, t\right) \right)$$
$$\Rightarrow \phi^* \frac{\partial \phi}{\partial t} - \frac{i}{2m} \phi^* \nabla^2 \phi = 0$$
$$(-\phi) \cdot \left(i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m} \phi\left(\vec{x}, t\right) \right)$$
$$\Rightarrow -\phi \frac{\partial \phi^*}{\partial t} - \frac{i}{2m} \phi \nabla^2 \phi^* = 0$$

By subtracting both results:

$$\Rightarrow \underbrace{\left(\phi^* \frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial t} \right)}_{\frac{\partial}{\partial t}(\phi^*\phi) = \frac{\partial}{\partial t}|\phi|^2 = \frac{\partial}{\partial t}\rho} - \frac{i}{2m} \underbrace{\left(\phi^* \nabla^2 \phi - \phi \nabla^2 \phi^* \right)}_{\vec{\nabla}\vec{j}} = 0$$

using continuity equation:

$$\vec{j} = \frac{-i}{2m} \left(\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^* \right)$$

2.2 Relativistic particles: Klein-Gordon Equation

Schrödinger equation:

1. Order derivative in time
2. Order space deriative

$\Rightarrow$ Is not covariant (time & space treated differently). Same approach but relativistic energy-momentum relation:

$$E^2 = \vec{p}^2 + m^2 \quad E- \rightarrow i\hbar\frac{\partial}{\partial t} \;;\; \vec{p} \rightarrow -i\hbar\vec{\nabla}$$
$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(\vec{x},t) = 0$$

four-vector notation:

$$p^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \rightarrow \begin{pmatrix} i\frac{\partial}{\partial t} \\ -i\vec{\nabla} \end{pmatrix} = i\partial^\mu$$

Definition 2.3. Klein-Gordon Equation

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi = 0$$

equation of motion of wavefunctions of relativistic particles (with spin 0). Manifestly lorentz-invariant (consists only of lorentz-invariants).

Definition 2.4. Continuity equation

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\vec{j} = 0$$
$$\Rightarrow \partial_\mu j^\mu = 0 \quad j^\mu = (\rho, \vec{j})$$

also lorentz-invariant.

How do $\rho, \vec{j}$ depend on ϕ?

$$\rho = i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\vec{j} = -i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)$$

Four-vector:

$$j^\mu = i\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*$$

probability density current. Fundamental free particle solution

$$\phi(\vec{x},t) \equiv \phi(x) = N \cdot e^{i(\vec{p}\vec{x}-Et)}$$
$$= Ne^{-p_\mu x^\mu} = Ne^{-ipx}$$

$$j^\mu = 2p^\mu \left|N\right|^2$$

Probability density: 0. component:

$$\rho = 2E\left|N\right|^2 \sim E$$

Why? Localisation probability $\rho\,\mathrm{d}^3x$ must be invariant under lorentz-transformation:

$$\mathrm{d}^3x \to \mathrm{d}^3x \cdot \gamma^{-1}$$

due to contraction of volume along the boost. $\rho\,\mathrm{d}^3x$ is invariant:

$$\rho \to \rho\gamma$$

i.e. ρ transforms like an energy under lorentz-transformation ($E \to \gamma E$).

Energy eigenvalues of free particle solution of Klein-Gordon-equation?

$$\begin{aligned}
\left(\partial_\mu\partial^\mu + m^2\right)\phi(x) &= 0\\
\Rightarrow \left(\partial_\mu\partial^\mu + m^2\right)e^{-ipx} &= 0\\
\left((-i)^2 p_\mu p^\mu + m^2\right)e^{-ipx} &=\\
-\underbrace{p_\mu p^\mu}_{E^2-|\vec{p}|^2} - m^2 &= 0\\
E^2 &= \vec{p}^2 + m^2\\
E &= \pm\sqrt{|\vec{p}|^2 + m^2}
\end{aligned}$$

since $\rho \sim E$ one has solution $\rho < 0$ negativ energy, negative probabilty density? $\to$ Unphysical. One way out: Feyman-Stückelberg approach: Interpret j^μ as a charge current. Thus ρ becomes charge density which can be negative:

$$j^\mu \to j^\mu = q \cdot j^\mu$$

or an electron ($q = -e$):

$$j^\mu(e^-) = -ie\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$$

Free particle solution ($E > 0$):

$$j^\mu(e^-) = -2ep^\mu\left|N\right|^2 = -2e\left|N\right|^2\begin{pmatrix}E\\ \vec{p}\end{pmatrix}$$

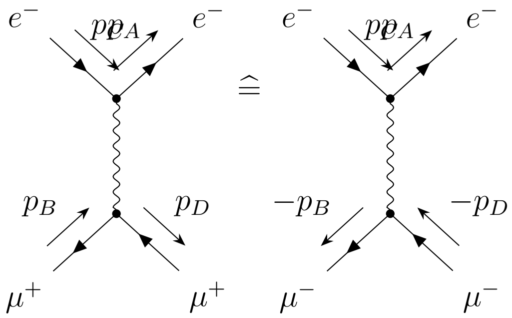
Negative energy solution: Consider an anti-muon with $E > 0$:

$$j^\mu(\mu^+) = +2ep^\mu\left|N\right|^2 = +2e\left|N\right|^2\begin{pmatrix}E\\ \vec{p}\end{pmatrix}$$

$$= \underbrace{-2e\,|N|^2 \begin{pmatrix} -E \\ -\vec{p} \end{pmatrix}}_{\text{muon with } E<0 \text{ , moving backwards}}$$

$$= -j^{\mu}(\mu^-)$$

→ Solutions for particles with negative energy can be used to describe antiparticles with positive energy.



Consequence: In diagrams involving antiparticles use particle state with $p^\mu \to -p^\mu$ for calculations.

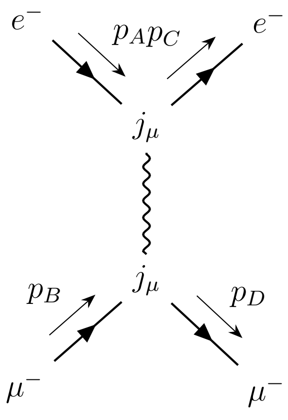
Convention: Arrows in feynman diagrams show direction of particle-current.

2.3 Crossing Symmetry

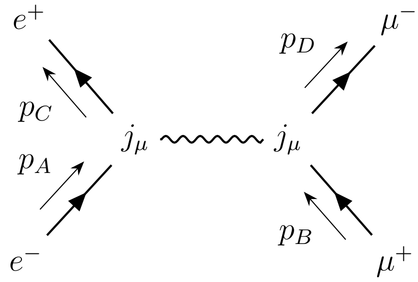
Amplitudes of Feynman diagrams are highly symmetric with respect to reversal or interchange of time and space.

→ consequence of time reversal symmetry and fact that amplitudes can be written in covariant form (i.e. space and time are treated equally)

Example 2.1. $e^-\mu^-$ scattering (QED)

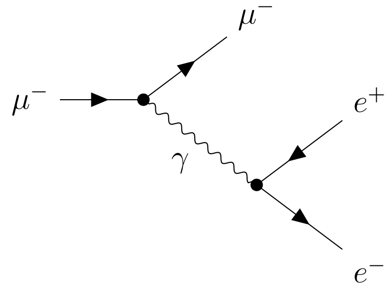


space-time crossing:



(Two more rotations)

Exchange f incoming μ^+ by outgoing μ^-



All these processes share one common transition amplitude, as they contain the same basic interaction. Only difference: Momenta of external particles.

→ Once amplitude of one process is known it is easy to calculate the other by substitution of four momentum. Amplitude for process AB → CD can be written in terms of Lorentz-invariant Mandestam-variables:

$$s = (p_A + p_B)^2$$
$$t = (p_A - p_C)^2$$
$$u = (p_A - p_D)$$

with $srt + u = m_A^2 + m_B^2 + m_C^2 m_D^2$. Crossing a process generally means interchange of s, t, u.

2.4 Interaction of a particle with a potential

Schrödinger equation of particle in potential $V(\vec{x}, t)$

$$i\frac{\partial\Phi\left(\vec{x},t\right)}{\partial t}\left(\underbrace{H_0}_{\text{Hamiltonian of free parcle}} + V\left(\vec{x},t\right)\right)\Phi(\vec{x},t)$$
$$\rightarrow \Phi_n(\vec{x},t) = \Phi_n(\vec{x})e^{-iE_nt}\ ,\ E_n = \text{const}$$

Potential V allows for transitions $\Phi_i \to \Phi_f$ (an interaction). Transition amplitude:

$$T_{fi} = \langle \Phi_f | V | \Phi_i \rangle$$

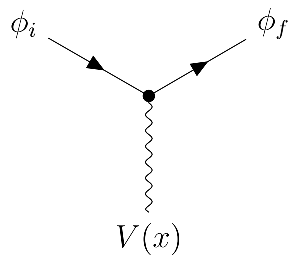
Assume V is small and acts only on short time scale T.

$\to$ Φ_i, Φ_i can be treated as plane waves with known energies E_i, E_f.

$\to$ V can be expanded in a perturbation series. First-order expansion (Born approximation):

$$T_{fi} = -i \int \mathrm{d}^4 \Phi_f^*(x) V(x) \Phi_i(x)$$

Corresponding to tree-level diagram:



Assume $V(x)$ is independent of time:

$$T_{fi} = -i \underbrace{\int \mathrm{d}^3 x \Phi_f^*(\vec{x}) V(\vec{x}) \Phi_i(\vec{x})}_{V_{fi}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-i(E_f - E_i)t} \, \mathrm{d}t}_{\text{time components of free particle wave functions}}$$
$$= -i V_{fi} 2\pi \underbrace{\delta(E_f - E_i)}_{\text{energy conservation}}$$

Transition probability (per unit time):

$$W_{fi} = \lim_{T \to \infty} \frac{|T_{fi}|^2}{T} = \lim_{T \to \infty} \frac{|V_{fi}|}{T} 2\pi \delta(E_f - E_i) \int_{-\infty}^{\infty} e^{-i(E_f - E_i)t} \, \mathrm{d}$$
$$= \lim_{T \to \infty} \frac{|V_{fi}|^2}{T} 2\pi \underbrace{\delta(E_f - E_i) \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-i(E_f - E_i)t} \, \mathrm{d}}_{T\delta(E_f - E_i)}$$
$$= 2\pi |V_{fi}|^2 \, \delta(E_f - E_i)$$

W_{fi} is proportional to cross section and decay rate. Initial state Φ_i usually known, but need to integrate over many possible final states Φ_f in the detector. Assume phase density $\rho_f = \rho_f(E_f)$:

$$W_{fi} \to W_{fi} = 2\pi \int_0^{\infty} \mathrm{d}E_f \rho_f(E_f) |V_{fi}|^2 \, \delta(E_f - E_i) = 2\pi |V_{fi}|^2 \, \delta_f(E_i)$$

(Fermi's golden rule)

Here: Non-relativistic treatment: We will see how this works for relativistic particle collision soon. Proper treatment of phase space only important for fermions (but looking at spin-0 particles here).

3 Electrodynamics of spinless particles

Apply calculation of tree-level amplitude to electromagnetic potential.

1. Interaction of charged particle in such potential
2. Interaction in potential provided by other particle ($\rightarrow$ scattering)

3.1 Covariant Electrodynamics

Maxwell's equations:

- $\vec{\nabla} \cdot \vec{E} = \rho$
- $\vec{\nabla} \cdot \vec{B} = 0$
- $\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$
- $\vec{\nabla} \times \vec{B} = \vec{j} + \dot{\vec{E}}$

four coupled differential equations in fields $E(\vec{x}, t), B(\vec{x}, t)$. Express by potentials:

$$\vec{B} = \vec{\nabla} \times \vec{A}\vec{E} = -\vec{\nabla}\phi - \vec{A}$$

Freedom of gauge: Choose Lorentz-Gauge:

$$\vec{\nabla}\vec{A} + \dot{\phi} = 0$$

$\Rightarrow$ Two decoupled differential equations in $\phi, \vec{A}$:

$$\vec{\nabla}^2\vec{A} - \frac{\partial^2\vec{A}}{\partial t^2}\partial t^2 = -\vec{j}$$
$$\vec{\nabla} - \frac{\partial^2\phi}{\partial t^2} = -\rho$$

With $A^\mu \equiv (\phi, \vec{A})$ and $j^\mu = \binom{\rho}{\vec{j}}$:

$$\partial_\mu\partial^\mu A^\nu = j^\nu$$

Maxwell's equations in covariant form. Similarly: Lorentz gauge

$$\partial_\mu A^\mu$$

3.2 The "spinless electron" in an EM field

How to express presence of e.m. field potential in equation of motion? Classical electrodynamic (e.g. Jackson):

$$\begin{aligned} E &\to E - \varphi \cdot p \\ \vec{p} &\to \vec{p} - q\vec{H} \\ \Rightarrow p^\mu &\to p^\mu - qA^\mu \end{aligned}$$

for a particle of charge q in e.m. potential A^μ

$$\begin{aligned} E - q\varphi &\to i\frac{\partial}{\partial t} - q\varphi \\ \vec{p} - q\vec{A} &\to -i\vec{\nabla} - q\vec{A} \\ &\Rightarrow i\partial^\mu - qA^\mu \end{aligned}$$

in total, for electron with charge $q = e$

$$p^\mu \to i\partial^\mu + eA^\mu$$

insert in Klein-Gordon equation:

$$\begin{aligned} &(i\partial_\mu + eA_\mu)\,(i\partial^\mu + eA^\mu)\,\Phi(x) - m^2\phi(x)) = 0 \\ \to &\left(\partial_\mu\partial^\mu - \underbrace{ie\,(A_\mu\partial^\mu + \partial_\mu A^\mu) - e^2A^2}_{\equiv V_{em}(x)} + m^2\right)\phi(x) = 0 \end{aligned}$$

perturbation of the free particle Hamiltionian H_0 due to coupling of electron to A^μ. Interpreted in leading order approximation: neglect terms $\sim e^-$.

$$V_{em}(x) = -ie\,(\partial_\mu A^\mu + A_\mu\partial^\mu)$$

Leading order transition amplitude:

$$\begin{aligned} T_{fi} &= -i\int \mathrm{d}^4x\phi_f^*(x)V_{em}(x)\phi_i(x) \\ &= -i\int \mathrm{d}^4(-ie)\phi_f^*\,(\partial_\mu A^\mu + A_\mu\partial^\mu)\,\phi_i \end{aligned}$$

Integration by parts:

$$\int_{-\infty}^{\infty}\mathrm{d}^4x\,\underbrace{\phi_f^*}_{u}\,\underbrace{\partial_\mu A^\mu\phi_i}_{v'} = \underbrace{\underbrace{\phi_f^*}_{u}\underbrace{\sum_\mu A^\mu\phi_i}_{v}\Bigg|_{-\infty}^{\infty}}_{0\text{ because }A\to 0\text{ for }x\to\infty} - \int_{-\infty}^{\infty}\mathrm{d}^4x\,\underbrace{(\partial_\mu\phi_f^*)}_{u'}\,\underbrace{A^\mu\phi_i}_{v}$$

$$= - \int_{-\infty}^{\infty} \mathrm{d}^4 x \left(\partial_\mu \phi_f^*\right) A^\mu \phi_i$$

hence:

$$T_{fi} = -i \int \mathrm{d}^4 x (-ie) \left[-\partial_\mu \phi_f^* A^\mu \phi_i + \phi_f^* A^\mu \partial_\mu \phi_i\right]$$

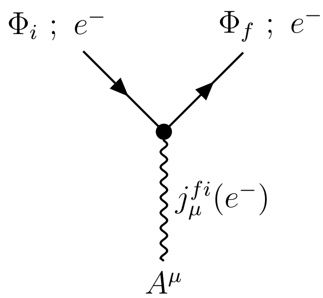
$$= -i \int \mathrm{d}^4 x \underbrace{(- \underbrace{ie}_{\text{coupling constant}}) \left[\phi_f^* \partial_\mu \phi_i - \left(\partial_\mu \phi_f^*\right) \phi_i\right]}_{j_\mu^{fi}:\ \text{Four-vector current of } e^-} \underbrace{A^\mu}_{\text{interaction four-potential}}$$

so finally

$$T_{fi} = -i \int d^4 j_\mu^{fi} A^\mu$$

four vector density current of the electron

$\rightarrow$ coupling between the electron charge current and the e.m. potential with coupling strength e.



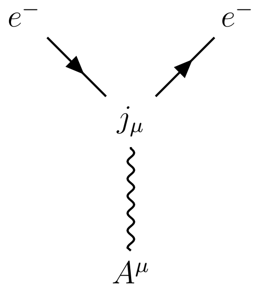
Remember: We work in a free-particle approximation

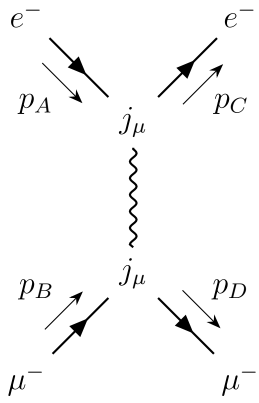
$$\phi_i(x) = N_i e^{-ip_i x}$$
$$\phi_f(x) = N_f e^{-ip_f x}$$
$$\Rightarrow j_\mu^{fi}(x) = e N_i N_f \left(p_i + p_f\right)_\mu e^{-i(p_i - p_f)x}$$

charge density current of scattering electron in free-particle approximation.

3.3 Spinless electron-muon scattering

E.m. potential A^μ now provided by a muon current.



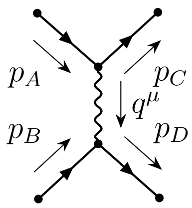


Connection between potential A^μ and muon current $j_\mu^{(2)}$:

$$\partial_\mu \partial^\mu \underbrace{A^\nu}_{\text{potential provided by ...}} = \underbrace{j^\nu_{(2)}}_{\text{muon current}} \qquad \text{(Maxwell)}$$

Charge density current of muon (free-particel approximation):

$$j^\mu_{(2)} = -\partial_\mu \partial^\mu A^\nu = eN_B N_D (p_B + p_D)^\mu e^{-i\underbrace{(p_B - p_D)}_{\equiv -q}x}$$



$$q^\mu = (p_A - p_C)^\mu = (p_D - p_B)^\mu$$

obviously, $q^2 \neq 0 = m_j^2$ (rel. energy-momentum relation).

Solution of Klein-Gordon equation:

$$A^\mu(x) = - \underbrace{\frac{1}{q^2}}_{\text{decreases strongly with momentum transfer}} j^\mu_{(2)}(x)$$

e.m. of a fly-by muon. → not a “real” photon which can freely propagate in space, but a “virtual” photon. Possible because of uncertainty principle. Transition amplitude in leading order:

$$T_{fi} = -i \int \mathrm{d}^4 x j_\mu^{(1)}(x) A^\mu(x)$$

$$= -i \int \mathrm{d}^4x \; \underbrace{j_\mu^{(2)}}_{\substack{\text{electron}\\\text{current}}} (x) \underbrace{\left(-\frac{1}{q^2}\right)}_{\substack{\text{momentum}\\\text{transfer}}} \underbrace{j_\mu^{(1)}}_{\substack{\text{muon}\\\text{current}}} (x)$$

Integration similar to 2.4 in plane-wave approximation:

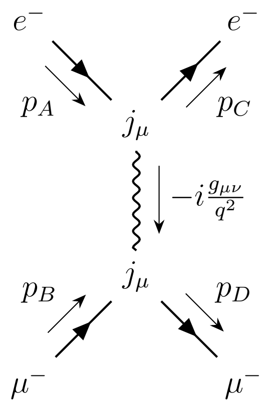
$$T_{fi} = \underbrace{-iN_A N_B N_C N_D}_{\substack{\text{wave function}\\\text{normalisation}}} (2\pi)^4 \underbrace{\delta^4\left(p_D + p_C - p_B - p_A\right)}_{\substack{\text{component-wise energy}\\\text{momentum conservation}}} \cdot \mathcal{M}$$

with the invariant amplitude $\mathcal{M}$ given by

$$-i\mathcal{M} = \underbrace{\left(ie\left(p_A + p_C\right)^\mu\right)}_{e^- \text{with coupling constan } e} \cdot \underbrace{\left(-i\frac{g_{\mu\nu}}{q^2}\right)}_{\text{photon propagator}} \cdot \underbrace{\left(ie\left(p_B + p_D\right)^\nu\right)}_{\text{muon with coupling constant } e}$$

$\mathcal{M}$ contains the physics of the interaction process. Why $g_{\mu\nu}$? No explanation here, but due to fact that photon is boson with spin$= 1$.

Feynman diagram for $-i\mathcal{M}$:



1. Vertex: $ie(p_A + p_C)^\mu$
2. Vertex: $ie(p_B + p_D)^\nu$

3.4 Towards the electron-muon to positron anti-muon cross section

Wave function normalization:

$$\phi(x) = Ne^{-ipx}$$

prob. density: $\rho = |\phi^2| = 2E\,|N|^2$ Probability to find particle in (large, but arbitrary) volume V:

$$\int_V \mathrm{d}^3x\rho \stackrel{!}{=} 1 \to N = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2E}}$$

Transition rate: Probability per unit "volume-time"

$$W_{fi}\frac{|T_{fi}|}{TV} = \frac{1}{TV}\left|-iN_A N_B N_C N_D (2\pi)^4 \delta^4(p_C + p_D - p_A - p_B) M\right|^2$$

$$= \frac{1}{TV} \frac{1}{16V^4 E_A E_B E_C E_D} (2\pi)^4 \, TV \delta^4(\dots) \, |M|^2$$

Cross section: AB→CD, experimental in the lab system

Scattering:

$$\mathrm{d}N_A = -N_A \frac{\mathrm{d}x}{\lambda} = -N_A \underbrace{n_B}_{\substack{\text{target-}\\ \text{density}}} \underbrace{\sigma}_{\substack{\text{cross-}\\ \text{section } [\mathrm{m}^2]}} \mathrm{d}x \to N_A(\Delta x) = \mathrm{N}_A(0) e^{-n_B \sigma \Delta x} \simeq N_A(0)(1 - n_B \sigma \Delta x)$$

e.g. number of particles absorbed = number of particles scattered. Number of particles scattered:

$$\underbrace{N_C(\Delta x)}_{\text{measured}} = \underbrace{N_A(0) n_B \Delta x \sigma}_{\text{integrated luminosity, } [L]=/\mathrm{m}^2}$$

$$\sigma = \frac{N_C}{L} \text{ or } \frac{\mathrm{d}\sigma}{\mathrm{d}\Sigma} = \frac{1}{L} \frac{\mathrm{d}N_C}{\mathrm{d}\Sigma}$$

What is the connection to W_{fi}?

$$W_{fi} = \frac{|T_{fi}|^2}{TV} = \frac{N_C}{TV}$$

Consider interaction of A, B in volume $V = F \cdot \Delta x$ with interaction time $\Delta T = \frac{\Delta x}{|\vec{v_A}|}$:

$$\sigma = \frac{N_C}{L} = \frac{N_C}{N_A \cdot m_B \cdot \Delta x} = \frac{N_C}{\Delta T V} \cdot \frac{\Delta T}{\Delta x} \cdot \frac{V}{N_A} \frac{1}{n_B}$$

$$= W_{fi} \cdot \frac{1}{|\vec{v_A}|} \frac{1}{n_A} \frac{1}{n_B} \cdot n_{\text{final states}}$$

$$\Rightarrow \sigma = \frac{W_{fi}}{\underbrace{\vec{v_A} n_A}_{\substack{\text{flux density}\\ \text{of A } [1/\mathrm{m}^2 s]}} \; n_B} \cdot n_{\text{final states}}$$

with $n_A = n_B = \frac{1}{V}$ (one particle in the volume). Number of final states? Assuming fermions, each final state needs a phase state volume $h^3 = (2\pi\hbar)^3$ in the 6 dimensional phase space $\mathrm{d}^3x \, \mathrm{d}^3p = V \, \mathrm{d}^3P$. Only a certain amount of numbers of states is allowed in the final phase space (quantum-mechanically allowed).

$$\text{Number of states } = \frac{V \, \mathrm{d}^3p}{(2\pi\hbar)^3} \cdot \underbrace{\frac{1}{\text{no. of particles in } V}}_{\substack{n_A = n_B = 1\\ \text{wave function normalisation}}} = \frac{V \, \mathrm{d}^3p}{(2\pi)^3} \quad (\hbar = 1)$$

Differential crossection for scattering into momentum elements $\mathrm{d}^3p_C \, \mathrm{d}^3p_D$ around p_C, p_D

$$\boxed{\mathrm{d}\sigma = \frac{W_{fi}}{|\vec{v_A}|^2} \frac{V \, \mathrm{d}^3p_C}{(2\pi)^3} \cdot \frac{V \, \mathrm{d}^3p_D}{(2\pi)^3}}$$

(One phase space factor per each final state particel)

$$\mathrm{d}\sigma = \underbrace{\frac{1}{|\vec{v_A}|\, 2E_A 2E_B}}_{\text{flux factor}} \cdot \underbrace{|\mathcal{M}|^2}_{\substack{\text{invariant}\\ \text{amplitude}}} \underbrace{(2\pi)^4 \delta^4(p_C + p_D - p_B - p_A) \cdot \frac{\mathrm{d}^3 p_C}{(2\pi)^3 2E_C} \cdot \frac{\mathrm{d}^3 p_D}{(2\pi)^3 2E_D}}_{\mathrm{d}Q \text{Lorentz invariant phase space } (= \text{DLIPS})}$$

$$\mathrm{d}Q = (2\pi)^4 \delta^4 \left(p_C + p_D - p_A - p_B\right) \cdot \frac{\mathrm{d}^3 p_C}{(2\pi)^3 2E_C} \cdot \frac{\mathrm{d}^3 p_D}{(2\pi)^3 2E_D}$$

invariant phase space factor. In short:

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{F} \,\mathrm{d}Q$$

What is F? If B is stationary, then for the Flux-Factor

$$F = |\vec{v_A}| \cdot 2E_A 2E_B$$

In general:

$$F = |\vec{v_A} - \vec{v_B}|\, 2E_A 2E_B$$

for collinear collision $(|\vec{v_A} - \vec{v_B}|) = |\vec{v_A}| + |\vec{v_B}|$

$$\begin{aligned} F &= |\vec{v_A} - \vec{v_B}|\, 2E_A 2E_B \\ &= (|\vec{v_A}| + |\vec{v_B}|)\, 4E_A E_B \\ &= 4\left(|\vec{p_A}| \cdot E_B + |\vec{p_B}|\, E_A\right) \\ &= 4\left[(p_A p_B)^2 - m_A^2 m_B^2\right]^{\frac{1}{2}} \end{aligned}$$

For a general collinear collision:

$$F = 4\left[(p_A p_B)^2 - m_A^2 m_B^2\right]^{\frac{1}{2}}$$

relatively easy to show that in centre of mass frame: $\vec{p_A} = -\vec{p_B}$.

$$\boxed{F = 4\,|\vec{p_A}|\,\sqrt{s} = 4\,|\vec{p_i}|\,\sqrt{s}}$$

$$s = (E_A + E_B)^2 \qquad |\vec{p_i}| = |\vec{p_A}| = |\vec{p_B}|$$

Invariant phase space factor: Want to obtain $\frac{\mathrm{d}\sigma}{\mathrm{d}\theta}(\theta)$: $\mathrm{d}^3 p_C\, \mathrm{d}^3 p_D \to \mathrm{d}\Omega$. Measure in $\mathrm{d}\Omega$ and not in $\mathrm{d}^3 p_C$ and $\mathrm{d}^3 p_D$. Momentum conservation: $p_D = (E_d, \vec{p_D})$ fully determined by p_A, p_B, p_C.

- Not interested in particle D (no need to detect it)
- Integrate $\mathrm{d}Q$ over $\mathrm{d}^3 p_D$

$$\mathrm{d}Q \to \mathrm{d}Q = \frac{1}{4\pi^2}\frac{\mathrm{d}^3 p_C}{2E_C}\int \frac{\mathrm{d}^3 p_D}{2E_D}\delta^4(p_D + p_C - p_A - p_B)$$

$$\delta^4(\dots) = \delta(E_D + E_C - E_A - E_B)\cdot\delta^3(\vec{p_D} + \vec{p_C} - \vec{p_A} - \vec{p_B})$$

$$\mathrm{d}Q = \frac{1}{4\pi^2}\frac{\mathrm{d}^3 p_C}{2E_C}\frac{1}{2E_D}\delta(E_A + E_B - E_C - E_D)$$

What about $\mathrm{d}^3 p_C$? In spherical coordinates:

$$\mathrm{d}^3 p = p^2\,\mathrm{d}p \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = p^2\,\mathrm{d}p\,\mathrm{d}\Omega$$

$$\mathrm{d}^3 p_C = |\vec{p_f}|^2\,\mathrm{d}\,|p_F|\,\mathrm{d}\Omega$$

Not interested in momentum of particle C since energy and scattering angle are related $\Rightarrow$ with $|\vec{p_f}| = |\vec{p_C}| = |\vec{p_D}|$. Integrate over $\mathrm{d}\,|\vec{p_f}|$:

$$\mathrm{d}Q = \frac{1}{4\pi}\frac{|p_f|^2\,\mathrm{d}\,|p_f|}{4E_C E_D}\,\mathrm{d}\Omega\delta(\sqrt{s} - E_C - E_D)$$

Collision energy $\sqrt{s} = E_C + E_D (= E_A + E_B) = \sqrt{m_C^2 + p_f^2} + \sqrt{m_D^2 + p_f^2}$

$$\frac{\mathrm{d}(\sqrt{s})}{\mathrm{d}p_f} = 2\,|p_f|\cdot\left[\frac{1}{2\sqrt{m_C^2 + p_f^2}} + \frac{1}{2\sqrt{m_D^2 + p_f^2}}\right]$$

$$= |\vec{p_f}|\left(\frac{1}{E_C} + \frac{1}{E_D}\right) = |\vec{p_f}|\cdot\frac{E_C + E_D}{E_C\cdot E_D}$$

Put into $\mathrm{d}Q$:

$$\mathrm{d}Q = \frac{1}{4\pi^2}\frac{|\vec{p_f}|^2}{4E_C E_D}\frac{E_C E_D}{|\vec{p_f}|\,(E_C + E_D)}\,\mathrm{d}\Omega\,\mathrm{d}(\sqrt{s})\delta\left(\sqrt{s} - E_C - E_D\right)$$

Integrate over $\mathrm{d}(\sqrt{s})$

$$\mathrm{d}Q \to \mathrm{d}Q = \frac{1}{4\pi^2}\cdot|\vec{p_f}|\,\mathrm{d}\Omega\int\frac{1}{4\sqrt{s}}\delta()\,\mathrm{d}(\sqrt{s})$$

$$\mathrm{d}Q = \frac{1}{4\pi^2}\frac{|\vec{p_f}|}{4\sqrt{s}}\,\mathrm{d}\Omega$$

in the CMS. Differential cross section:

$$\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{F}\,\mathrm{d}Q = \frac{|\mathcal{M}|^2}{4\,|\vec{p_i}|\,\sqrt{s}}\frac{|p_f|}{4\sqrt{s}}\frac{1}{4\pi^2}\,\mathrm{d}\Omega$$

Differential electron-muon scattering cross section in the CMS. Total cross-section:

$$\sigma = \int_\Omega \mathrm{d}\Omega\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$$

for electron-muon scattering in CMS:

$$-i\mathcal{M} = ie(p_A + p_C)^\mu \left(-\frac{i}{q^2}\right) ie(p_B + p_D)_\mu = \frac{ie^2}{q^2}(p_A + p_C)^\mu (p_B + p_D)_\mu$$

high energy limit (neglect particle masses).

$$p_A = \begin{pmatrix} E_A = |\vec{p_i}| \\ \vec{p_i} \end{pmatrix}, p_B = \begin{pmatrix} |\vec{p_i}| \\ -\vec{p_i} \end{pmatrix}, p_C = \begin{pmatrix} |\vec{p_f}| \\ -\vec{p_f} \end{pmatrix}, p_D = \begin{pmatrix} |\vec{p_f}| \\ -p_f \end{pmatrix}$$

and use $q = p_D - p_B$ with above equations:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{CMS}} = \frac{\alpha^2}{4s}\left(\frac{s-u}{t}\right)^2 = \frac{\alpha^2}{4s}\left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2$$

for spinless electron-muon scattering.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim \frac{(\text{coupling constant})^{2\cdot\text{no. of vertices}}}{(\text{collision energy})^2}$$

Can use calculation to obtain cross-section for $e^+e^- \to \mu^+\mu^-$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(\frac{t-u}{s}\right)^2 = \frac{\alpha^2}{4s}\cos\theta$$

spinless $e^+e^- \to \mu^+\mu^-$ annihilation/pair-production.

3.5 Calculating decay rates

Differential decay rates for proesses

$$A \to 1 + 2 + 3 + \cdots + n$$

$$\mathrm{d}\Gamma = \frac{1}{2E_A}|\mathcal{M}|^2 \cdot \underbrace{\frac{\mathrm{d}^3p_1}{(2\pi)^3 2E_1} \cdot \frac{\mathrm{d}^3p_2}{(2\pi)^3 2E_2} \cdot \cdots \cdot \frac{\mathrm{d}^3p_n}{(2\pi)^3 2E_n}}_{\text{phase space factor for } n \text{ final state particles}} \cdot (2\pi)^4 \delta^4(p_A - p_1 - p_2 - \cdots - p_n)$$

$$= \frac{1}{2E_A}|\mathcal{M}|^2 \,\mathrm{d}Q$$

Compare that to interaction rate $\mathrm{d}\sigma = \frac{|\mathcal{M}|^2}{F}\,\mathrm{d}Q$. For colinear collisions: $F = 4|\vec{p_i}|\sqrt{s}$.
⇒ Similar to electron-muon-scattering but different flux-factor. For a two-body decay ($A \to 1 + 2$) in the CMS using $\mathrm{d}Q = \frac{1}{4\pi^2}\frac{|p_f|}{4\sqrt{s}}\,\mathrm{d}Q$:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} = \frac{|\vec{p_f}|}{32\pi^2 m_A^2} \cdot |\mathcal{M}|^2$$

$$|\vec{p_f}| = |\vec{p_1}||\vec{p_2}| \quad s = (p_A + p_B)^2 = p_A^2 = m_A^2$$

Total decay rate: Integrate $\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega}$ over full solid angle:

$$\tau = \Gamma^{-1}$$

is the liftime of the particle A, if only on decay channel exists. Otherwise for n channels:

$$T_{\text{tot}} = \sum_i^n \Gamma_i \to \tau = \frac{1}{\Gamma_{\text{tot}}}$$

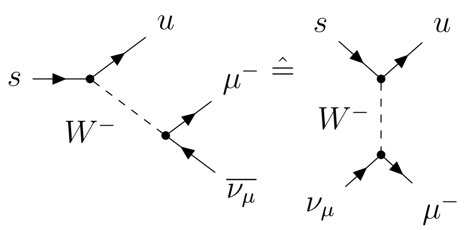
and branching rations

$$\text{BR}_i = \frac{\Gamma_i}{\Gamma_{\text{tot}}}$$

Remark. $\Gamma \sim |\vec{p_f}|$: When decay into different final states possible the one with the smallest mass (i.e. largest $|\vec{p_f}|$ preffered (phase space) if the rest of physics stays the same ($|\mathcal{M}|$).

How to calculate $|\mathcal{M}|$

Example 3.1. Weak decay



We know how to calculate scattering amplitude. Need proper coupling and W propagator which will be derived later.

4 Particles with spin $\frac{1}{2}$. The Dirac-equation

Long thought that the Klein-Gordon equation is the only relativistic generalization of the Schrödinger equation. Dirac found a wave equation that is linear in the time derivative.

→ Avoid negative probability densities found by Klein-Gordon. Should be covariant → linear in spatial derivatives.

Ansatz:

$$H\psi = (\vec{\alpha} \cdot \vec{p} + \beta \cdot m)\,\psi$$

$$H = i\frac{\partial}{\partial t}, \vec{p} = -i\vec{\nabla}$$

$$\left[i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi\right]$$

Requirement: ψ should fulfill the Klein-Gordon equation, i.e. $E^2 = \vec{p}^2 + m^2$ holds.

$$\rightarrow H^2\psi \stackrel{!}{=} \left(\vec{p}^2 + m^2\right)\psi$$
$$\rightarrow H^2\psi = (\vec{\alpha}\vec{p} + \beta m)(\vec{\alpha}\vec{p} + \beta m)\psi$$
$$= \left[\sum_{i=3}^{3}\left(\underbrace{\alpha_i^2}_{1} p_i^2 + \sum_{j<i}\underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{0} p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0} p_i m\right) + \underbrace{\beta^2}_{1} m^2\right]\psi$$
$$\stackrel{!}{=} (\vec{p}^2 + m^2)\psi$$
$$\Rightarrow \alpha_i^2 = 1 \forall \left(\alpha_x^2 = \alpha_y^2\alpha_z^2 = 1\right)$$
$$\beta^2 = 1$$
$$\{\alpha_i, \alpha_j\} = 0 \quad i \neq j$$
$$\{\alpha_i, \beta\} = 0$$
$$\Rightarrow \alpha, \beta \text{ cannot be simple scalars}$$

$\rightarrow \alpha_1, \alpha_2, \alpha_3, \beta$ must be (at least) 4x4 traceless Hermitian matrices.

1. $$\mathrm{Tr}(\alpha_j) = \mathrm{Tr}(\alpha_j \underbrace{\beta\beta}_{=1}) = \mathrm{Tr}(\beta\alpha_j\beta) = \mathrm{Tr}(-\beta\beta\alpha_j) = -\mathrm{Tr}(\beta\beta\alpha_j) = -Tr(\alpha_j)$$

2. Eigenvalues of $\alpha_j, \beta = \pm 1$
3. Because trace = sum of eigenvalues $\rightarrow \alpha_j, \beta$ must have even dimension
4. Because Dirac Hamiltonian must be Hermitian to have real eigenvalues, α_j, β must also be Hermitian

One possible representation is the Dirac-Pauli Representation:

$$\alpha_1 = \begin{pmatrix} \mathbf{0}_2 & \sigma_1 \\ \sigma_1 & \mathbf{0}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$\alpha_2 = \begin{pmatrix} \mathbf{0}_2 & \sigma_2 \\ \sigma_2 & \mathbf{0}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$
$$\alpha_3 = \begin{pmatrix} \mathbf{0}_2 & \sigma_3 \\ \sigma_3 & \mathbf{0}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} \mathbb{1}_2 & \sigma_2 \\ \sigma_2 & -\mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where $\sigma_{1,2,3}$ are the Pauli spin matrices. Properties:

- $\text{Tr}\beta = \text{Tr}\alpha_i = 0 \ \ \forall i$
- $\beta^\dagger = \beta \ , \ \alpha_i^\dagger = \alpha_i$

Consequence: Wave-function ψ is now a 4-dimensional vector, called a Dirac Spinor (not a Lorentz 4-vector!). Each element of the spinor ψ separately satisfies the Klein-Gordon equation (by construction)

$$\psi(\vec{r}, t) = \begin{pmatrix} \psi_1(\vec{r}, t) \\ \psi_2(\vec{r}, t) \\ \psi_3(\vec{r}, t) \\ \psi_4(\vec{r}, t) \end{pmatrix}$$

Later: Spinor describes particle/antiparticle states with two spin directions each.

4.1 Dirac-Equation on covariant form

So far:

$$i\frac{\partial}{\partial t}\psi = \left(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m\right)\psi$$

Multiply by β from the left

$$i\beta\frac{\partial}{\partial t}\psi = -i\beta\vec{\alpha}\vec{\nabla}\psi + \underbrace{\mathbb{1}_4}_{\beta\beta=\beta^2=\mathbb{1}_4} \cdot m\psi$$

Define a "four-vector" of matrices $\gamma^{0,1,2,3}$ as

$$\gamma^\mu = \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix} := \begin{pmatrix} \beta \\ \beta\alpha_1 \\ \beta\alpha_2 \\ \beta\alpha_3 \end{pmatrix} = \begin{pmatrix} \beta \\ \beta \cdot \vec{\alpha} \end{pmatrix}$$

$$\Rightarrow (i\gamma^\mu \cdot \partial_\mu - m)\psi = 0$$

the covariant form of the Dirac-equation, a set of four coupled differential equations. In Short notation using $\gamma^\mu \alpha_\mu =: \not{\alpha}$:

$$\left(i\not{\partial} - m\right)\psi = 0$$

or since $\hat{p}_\mu = i\partial_\mu$

$$\left(\not{p} - m\right)\psi = 0$$

Properties of he γ-matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & -\mathbb{1} \end{pmatrix} \quad , \quad \gamma^i \begin{pmatrix} \mathbf{0}_2 & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}$$

1. $(\gamma^0)^2 = \mathbb{1}_4$
2. $\left(\gamma^k\right)^2 = -\mathbb{1}_4 \quad k = 1, 2, 3$
3. $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \qquad \gamma \neq \nu$
4. $(\gamma^0)^\dagger = \gamma^0$ (Hermitian)
5. $\left(\gamma^k\right)^\dagger = -\gamma^k$ (anti-Hermitian)

In short

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \quad , \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$$

4.2 Probability an charge current density

Klein-Gordon: Add Klein-Gordon and complex-conjugated Klein-Gordon $\rightarrow$ continuity equation.

Dirac: Matrix-equation: Add Dirac-equation and the adjoint Dirac-equation with $\overline{\psi} = \psi^\dagger\gamma^0$. $\overline{\psi}$ is an adjoint spinor. Adjoint Dirac-equation:

$$i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = 0$$

From Dirac-equation and adjoint Dirac-equation can show that

$$\partial_\mu \underbrace{\left(\overline{\psi}\gamma^\mu\psi\right)}_{=j^\mu} = 0$$

the continuity equation. By multiplication:

$$j^\mu = \overline{\psi}\gamma^\mu\psi$$

the (conserved) probability density current.

4.3 Solutions to Dirac's equations

Solution is called a spinor. Plane wave ansatz: $\psi(x) = u(p)e^{-ipx}$ in particular $u(p)$ (also a spinor) only depends on $(E, \vec{p}$ not in x since $H = \vec{\alpha}\vec{p} + \beta m$ depends only on $(E, \vec{p})$. Dirac equation in covariant form:

$$\begin{aligned} \left(\gamma^\mu \partial_\mu - m\right) u(p)e^{-ipx} &= 0 \\ \Rightarrow \left(i\gamma^\mu \left(-ip_\mu\right) - m\right) u(p)e^{-ipx} &= 0 \\ \left(\gamma^\mu p_\mu - m\right) u(p) = 0 \text{ or } (\not{p} - m)u(p) &= 0 \end{aligned}$$

$\Rightarrow u(p)$ satisfies the Dirac equation. Which $u(p)$ solves this equation?

1. Look at particle at rest $p^\mu = \left(\begin{smallmatrix} E \\ \vec{0} \end{smallmatrix}\right)$
   $\psi = u(E, \vec{0})e^{-iEt}$
   $Hu = \beta m u = Eu$ (since $\vec{p} = 0$)
   $\gamma^\mu p_\mu = \gamma^0 p_0 = \gamma^0 E$

   $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot E \cdot u = m \cdot u$$

   4 independent solutions.

   $$\underbrace{u^{(1)}(\vec{p} = 0) = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad u^{(2)}(\vec{p} = 0) = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{with } E=+m}$$

   $$\underbrace{u^{(3)}(\vec{p} = 0) = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u^{(4)}(\vec{p} = 0) = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{with } E=-m}$$

   Full solution including the time dependence for a particle at rest is:

   $$\begin{aligned} \psi^{(1,2)}(\vec{p} = 0) &= e^{(1,2)}e^{-imt} \\ \psi^{(3,4)}(\vec{p} = 0) &= e^{(3,4)}e^{+imt} \end{aligned}$$

2. general solution for $\vec{p} \neq 0$:

   $$\left(\gamma^\mu p_\mu - m\right) u(p) = 0$$

   $$\Rightarrow \left[\underbrace{\begin{pmatrix} \mathbb{1}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbb{1}_2 \end{pmatrix}}_{\gamma^0 p_0} E - \underbrace{\begin{pmatrix} \mathbf{0}_2 & \vec{\sigma}\vec{p} \\ -\vec{\sigma}\vec{p} & \mathbf{0} \end{pmatrix}}_{\vec{\gamma}\vec{p}} \mathbb{1}_4 m \right] u(p) = 0$$

with $\vec{\sigma}\vec{p} = \sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$
Write in terms of $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$ ($u_A u_B$ are two column-vectors)

$$\begin{pmatrix} (E-m)\mathbb{1}_2 & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E-m)\mathbb{1}_2 \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$
$$\Rightarrow u_A = \frac{\vec{\sigma}\vec{p}}{E-m} u_B \qquad u_B = \frac{\vec{\sigma}\vec{p}}{E+m}$$
$$u_A = \frac{(\vec{\sigma}\vec{p})\,(\vec{\sigma}\vec{p})}{(E-m)\,(E+m)} u_A \qquad (\vec{\sigma}^2 = \mathbb{1})$$

Pick two orthogonal two-column vectos for u_A (arbitrary):

$$u_{A^{(1)}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to u_B^{(1)} = \left(\frac{1}{2+m}\right) \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$
$$\Rightarrow u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

with $E = +\sqrt{\vec{p}^2 + m^2} > 0$. Pick two orthogonal two-column vectors for u_B:

$$\Rightarrow u^{(1)} = N \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \qquad u^{(2)} = N \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

with $E = -\sqrt{\vec{p}^2 + m^2} < 0$

4.4 Spin

$u^{(1,2,3,4)}$ are degenerate in energy. Therefore there must be another observable (eigenvalue of an operator) that distinguishes $u^{(1)}$ from $u^{(2)}$ and $u^{(3)}$ from $u^{(4)}$.

$\Rightarrow$ Spin

Non-relativistic quantum mechanics:

$$\left[\hat{H}, \hat{\vec{L}}\right] = 0$$

$\Rightarrow$ Angular momentum is conserved.

Relativistic quantum mechanics:

$$\hat{H} = \vec{\alpha}\vec{p} + \beta m = -i\vec{\alpha}\vec{\nabla} + \beta m$$
$$\hat{\vec{L}}\vec{r} \times \vec{p} = i\vec{r} \times \vec{\nabla}$$

$$\Rightarrow \left[\hat{H}, \hat{\vec{L}}\right] = \left[\vec{\alpha}\hat{\vec{p}} + \beta m, \vec{r}\hat{\vec{p}}\right] = \left[\vec{\alpha}\hat{\vec{p}}, \vec{r} \times \hat{\vec{p}}\right]$$

$$\text{e. g. } \left[\hat{H}, \hat{\vec{L}}\right] = \left[\vec{\alpha}\hat{\vec{p}}, \left(\vec{r}\hat{\vec{p}}\right)_x\right] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]$$

$$= i(\alpha_y p_z - \alpha_z p_y)$$

$$\Rightarrow \left[\hat{H}, \hat{\vec{L}}\right] = -i(\vec{\alpha} \times \hat{\vec{p}}) \neq 0$$

"Orbital" angular momentum is not conserved.

$\Rightarrow$ There must be another operator that commutes with $\hat{H}$. Define spin operator.

$$\hat{\vec{S}} = \frac{1}{2}\hat{\vec{\Sigma}} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & \mathbf{0}_2 \\ \mathbf{0}_2 & \vec{\sigma} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Sigma_x \\ \Sigma_y \\ \Sigma_z \end{pmatrix}$$

e.g.

$$\left[H, \hat{S}_x\right] = \left[\alpha\vec{p} + \beta m, \frac{1}{2}\Sigma_x\right] = [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] = -i\left(\alpha_y p_z - \alpha_z p_y\right)$$

$$\Rightarrow \left[\hat{H}, \hat{\vec{S}}\right] = +i\left(\vec{\alpha} \times \vec{p}\right) \neq 0$$

$\Rightarrow$ Spin also not conserved. But for $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$: $\left[\vec{H}, \hat{\vec{J}}\right] = 0$

$\rightarrow$ Total angular momentum $\hat{\vec{J}}$ is conserved

$\rightarrow$ Interpret S as the intrinsic spin of the particle (not conserved!)

Magnitude of spin "derived" from Dirac equation. Like in quantum mechanics:

$$\hat{\vec{S}}^2 = \left(\frac{1}{2}\Sigma\right)^2 = \frac{1}{4}(\Sigma_x^2 + \Sigma_y^2 + \Sigma_z^2) = \frac{3}{4}\mathbb{1}_4$$

$$\Rightarrow \hat{\vec{S}}^2 u^{(i)} = s(s+1)u^{(i)} = \frac{3}{4}u^{(i)}$$

$$\Rightarrow S = \frac{1}{2}$$

Dirac equation gives spin-$\frac{1}{2}$ particles. Spin projection along a particular direction:

$$\hat{S}_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

for particles at rest:

$$\hat{S}_z u^{(1)}(\vec{p} = 0) = \frac{1}{2}u^{(1)}(\vec{p} = 0)$$

$$\Rightarrow S_z^{(1)} = +\frac{1}{2}$$

$$\hat{S}_z u^{(2)}(\vec{p}=0) = -\frac{1}{2}u^{(2)}(\vec{p}=0)$$

$$\Rightarrow S_z^{(2)} = -\frac{1}{2}\ ,\ S_z^{(3)} = +\frac{1}{2}\ ,\ S_z^{(4)} = -\frac{1}{2}$$

$u^{(1)}(\vec{p}=0)$ and $u^{(2)}(\vec{p}=0)$ are eigenstates of S_z but this is not generally true. However, if travelling in the z-direction then

$$u^{(1)} = N \begin{pmatrix} 1 \\ 0 \\ \pm\frac{p_z}{E+m} \\ 0 \end{pmatrix}$$

are eigenstates of S_z. Convenient to label the spinors by helicity rather than projection on a fixed axis:

$$h = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}$$

the spin projection onto the direction of motion. Operator:

$$\hat{h} = \frac{1}{|\vec{p}|}\hat{\vec{S}}\hat{\vec{p}} = \frac{1}{2|\vec{p}|}\left(\hat{\vec{\Sigma}}\vec{p}\right)$$

Can show that $\left[\hat{H},\hat{h}\right] = 0$. Thus can find spinors which are simultaneously eigenstates of energy and helicity.

Example 4.1. for $\vec{p} = p\hat{e_z}$:

$$2h = \frac{1}{|\vec{p}|}\vec{\Sigma}\vec{p} = \frac{1}{p}\Sigma_3 p =$$

$$\begin{pmatrix} \mathbf{0}_3 & \sigma_2 \\ \sigma_2 & \mathbf{0}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\hat{h}u^{(1)} = \lambda u^{(1)} \quad \lambda = +\frac{1}{2}$ (right handed)
$\hat{h}u^{(2)} = \lambda u^{(2)} \quad \lambda = -\frac{1}{2}$ (left handed)

$$\left.\begin{matrix} u^{(1)}.\ \overset{\vec{S}}{\Rightarrow}\!\!\!\!\to \vec{p} \\ u^{(2)}.\ \overset{\vec{S}}{\Leftarrow}\!\!\!\!\to \vec{p} \end{matrix}\right\} E > 0 \text{ Moving forward in time}$$

$$\left.\begin{matrix} u^{(3)}.\ \overset{\vec{S}}{\Leftarrow}\!\!\!\!\leftarrow \vec{p} \\ u^{(4)}.\ \overset{\vec{S}}{\Rightarrow}\!\!\!\!\leftarrow \vec{p} \end{matrix}\right\} E > 0 \text{ Moving backward in time}$$

More conflicted if $\vec{p}$ and $\vec{S}$ not aligned but still states one either right handed or left handed.

Remark. Helicity ist not a Lorentz-invariant. If the particle has mass you can always find a frame in which the particle is overtaken $\to$ momentum changes sign (spin stays the same) $\to$ helicity changes. The relativistic invariant is called chirality.

4.5 Antiparticles

How do we deal with the negative-energy solutions $u^{(3)}$ and $u^{(4)}$ that are travelling backwards in time? Charge density of the positron (in plane wave approximation)

$$j^\mu(e^+) = ie\overline{\psi}j^\mu\psi = +e2\begin{pmatrix} E \\ \vec{p} \end{pmatrix}$$
$$\left[\text{Plane wave: } \psi^{(i)}(x) = u^{(i)}(p)e^{-ipx}\right]$$
$$= -e2\begin{pmatrix} -E \\ -\vec{p} \end{pmatrix} = -j^\mu(e^-)$$

Feynman-Stückberg approach: Negative energy solutions can be used to describe anti particles with positive energy. For "physical energy" $E > 0, \vec{p} > 0$:

$$e^+(E, \vec{p}, \lambda) = e^-(-E, -\vec{p}, \lambda)$$

both $\vec{p} and \vec{S}$ change sign, so λ is unchanged.
Convention: Name antiparticles spinors differently, adopt $E > 0$, $\vec{p} > 0$ for all solutions to Dirac equation.

$$v^{(1)}(E, \vec{p}) = u^{(4)}(-E, -\vec{p})$$
$$v^{(2)}(E, \vec{p}) = u^{(3)}(-E, -\vec{p})$$

Subtle modification: Spin to "physical spin"

$$S_z^{(v)} = -S_z$$
$$u^{(4)} \overset{\Rightarrow}{\leftarrow} \overset{E,\vec{p}}{\rightarrow} \overset{\Leftarrow}{\rightarrow} \overset{\text{"physical spin"}}{\rightarrow} \overset{\Rightarrow}{\rightarrow} \lambda = +\frac{1}{2}$$
$$v^{(4)} : \overset{\Leftarrow}{\rightarrow} \lambda = -\frac{1}{2}$$

Using the antiparticle spinors implies changes to the Dirac equation.

$$\left(\not{p} - m\right) u(p) = 0 \qquad \text{For particle spinors}$$
$$\to \text{ solutions } u^{(1,2,3,4)}$$

For antiparticles (= particles with $E \to -E, \vec{p} \to -\vec{p}$):

$$\left(-\not{p} - m\right) u(-p) = \left(\not{p} + m\right) v(p) = 0$$

For antiparticle spinor $v(p)$.

Remark.

$$p^0 \equiv E > 0$$

Finally arrive at the solutions to the Dirac equation (with correct spinor normalization). Particles:

$$\psi^{(1,2)} = u^{(1,2)} \cdot e^{-ipx}$$

$$u^{(1)} = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \qquad \lambda = +\frac{1}{2}$$

$$u^{(1)} = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix} \qquad \lambda = -\frac{1}{2}$$

Antiparticles:

$$\psi^{(1,2)} = v^{(1,2)} e^{+ipx}$$

$$v^{(1)} = \sqrt{E+m} \begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \qquad \lambda = +\frac{1}{2}$$

$$u^{(2)} = \sqrt{E+m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m}1 \\ 0 \end{pmatrix} \qquad \lambda = -\frac{1}{2}$$

4.6 Charge conjugation and parity operator

Charge conjugation: Particles $\leftrightarrow$ antiparticle (swaps the charge(s) of the particle). Dirac-Pauli representation:

$$\psi_C = \hat{C}\psi = i\gamma^2\psi^*$$

$$\text{with } i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Example 4.2.

$$\psi = u^{(1)} e^{-ipx}$$

$$\psi_C = \hat{C}\psi = i\gamma^2 \left[u^{(1)}(p) e^{-ipx}\right] = \cdots = v^{(1)}(p) e^{+ipx}$$

Parity operator: Space inversion through the origin, i.e.:

$$(\vec{x}, t) \to (\vec{x}', t') = -(\vec{x}, t)$$

(Important because QED, QCD conserve parity).

$$\hat{P}^2 = \mathbb{1}$$

From this we can derive $\hat{P}$

$$\psi_P(x) = \hat{P}\psi(x) = \gamma^0\psi(x)$$

Example 4.3.

$$\begin{aligned}
\psi_P^{(1)}(x) &= \gamma^0\psi^{(1)}(x) = \gamma^0 u^{(1)}(p)e^{-i(Et-\vec{p}\vec{x})} \\
&= \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ -\frac{p_z}{E+m} \\ \frac{-p_x+ip_y}{E+m} \end{pmatrix} e^{-i(Et'-(-\vec{p})\vec{x})} \\
&= u^{(1)}(-\vec{p})e^{-i(Et'-(-\vec{p}\vec{x}'))} = \psi^{(1)}(\vec{x}', -\vec{p})
\end{aligned}$$

$\to$ Under space inversion:

- Momentum of particle swaps sign
- Spinor structure stays the same $\to$ no flip of sign, but helicity flips

$$\overset{\Rightarrow}{\rightarrow} \vec{p} \to \overset{\Rightarrow}{\leftarrow}$$

For particle at rest

$$\begin{aligned}
\hat{P}u^{(1,2)} &= \gamma^0 u^{(1,2)} = +u^{(1,2)} \\
\hat{P}v^{(1,2)} &= \gamma^0 v^{(1,2)} = -v^{(1,2)}
\end{aligned}$$

Particles: Positive intrinsic parity
Antiparticles: Negative intrinsic parity.

5 QED with spin-half particles

Want to scatter electron in electromagnetic potential A^μ. Replace

$$\begin{aligned}
p^\mu &\to p^\mu - qA^\mu \\
i\partial^\mu &\to i\partial^\mu - qA^\mu
\end{aligned}$$

For electron

$$i\partial^\mu \to i\partial^\mu + eA^\mu$$

Dirac equation

$$(i\gamma^\mu\partial^\mu + e\gamma^\mu A_\mu - m)\,\psi = 0$$

or

$$(i\gamma^\mu\partial_\mu - m)\,\psi = \gamma^0 V(x)\ \gamma^0 V(x) = -e\gamma^\mu A_\mu$$

or

$$V(x) = e\gamma^0\gamma^\mu A_\mu$$

e.g.

$$V_0(x) = -e\gamma^0\gamma^0 A_0 = -e\Psi$$

Transition probability (Born approx.)

$$\begin{aligned} T_{fi} = -i \int \mathrm{d}^4x\psi_f^\dagger(x)V(x)\psi_i(x) = \\ -ie \int \mathrm{d}^4x\psi_f^\dagger(x)\gamma^0\gamma^\mu A_\mu\psi_i \\ = ie \int \mathrm{d}^4x\overline{psi_f}\gamma^\mu A_\mu\psi_i = \\ ie \int \mathrm{d}^4 j^\mu_{fi} A_\mu \end{aligned}$$

with

$$j^\mu_{fi} = -e\overline{\psi_f}\gamma^\mu\psi_i$$

plane waves: $\psi(x) = u(p)e^{-ipx} \to \overline{\psi}(x) = \overline{u}(p)e^{+ipx}$

$$\to j^\mu_{fi} = -ie\overline{u_f}\gamma^\mu u_i e^{i(p_f-p_i)x}$$

Comparison to spinless case:

$$j^\mu_{fi,\,\text{spinless}} = -eN_iN_f \cdot (p_i + p_f)^\mu r^{i(p_f-p_i)x}$$

5.1 Feynman rules for QED

Full invariant amplitude for two-particle process: Write A_μ in terms of the e.m. current of the other particle. Building blocks (Feynman rules) for amplitude $-i\mathcal{M}$:

$$\left.\begin{array}{r}\rightarrow \cdot u(p) \\ \cdot \rightarrow \overline{u}(p)\end{array}\right\} \text{initial / final state particles (fermions)}$$

$$\left.\begin{array}{r}\leftarrow \cdot v(p) \\ \cdot \leftarrow \overline{v}(p)\end{array}\right\} \text{initial / final state antiparticles (fermions)}$$

Also possible with photons. The photon propagator:

$$-i\frac{g_{\mu\nu}}{q^2}$$

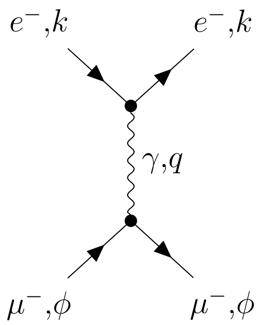
The fermion propagator:

$$-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$$

QED vertex:

$$-iQe\gamma^\mu$$

5.2 Electron-Muon scattering



Apply Feynman rules:

$$-i\mathcal{M} = \underbrace{(ie\overline{u}(k')\gamma^\mu u(k))}_{\text{electron current}} \underbrace{\left(-i\frac{g_{\mu\nu}}{q^2}\right)}_{\text{photon current}} \underbrace{(ie\overline{u}(p')\gamma^\nu u(p))}_{\text{muon current}}$$

$$= ie^2\overline{u}(k')\gamma^\mu u(k)\frac{1}{q^2}\overline{u}(p')\gamma_\mu u(p)$$

Most experiments use

- unpolarized beams → average over initial state helicity configurations
- spin-intensitive detectors → sum over final helicity configurations

Four initial states → 16 possible processes. Hence

$$|\mathcal{M}|^2 \to \overline{|\mathcal{M}|^2} = \frac{1}{(2S_A+1)(2S_B+1)} \sum_{\text{final state spin config}} |\mathcal{M}|^2$$

Use computer or trace techniques for calculation → Not here. Result:

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[(k'p')(kp) + (k'p)(kp') - m_e^2 p'p - m_\mu^2 k'k + 2m_e^2 m_\mu^2\right]$$

High energy limit ($m_e = m_\mu = 0$)

$$\overline{|\mathcal{M}|^2} 2e^4 \frac{s^2+u^2}{t^2}$$

invariant amplitude unpolarized electron-muon-scattering. Differential cross-section in CMS:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p_f}|}{\vec{p_i}}\overline{|\mathcal{M}|^2} = \frac{\alpha^2}{2s}\left[\frac{4+(1+\cos(\theta))^2}{(1-\cos(\theta))^2}\right]$$

electron-muon-cross-section in the CMS. Contributions:

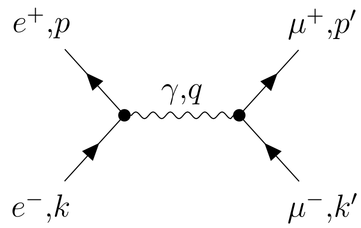
- $\frac{1}{(1-\cos(\theta))^2}$: Photon propagator $\cdot \frac{1}{q^2} = \frac{1}{(k-k')^2} = \frac{1}{t}$
  ⇒ Cross-section diverges as $q^2 \to 0$
- Helicity configurations: QED "conserves" helicity for each particle current. Four allowed helicity configurations.

$$\left.\begin{array}{l} e^-_\uparrow \mu^-_\uparrow \to e^-_\uparrow \mu^-_\uparrow \\ e^-_\downarrow \mu^-_\downarrow \to e^-_\downarrow \mu^-_\downarrow \end{array}\right\} J_z = 0 \text{ i.e. no preferred axis} \to \text{ isotropic distribution}$$

$$\left.\begin{array}{l} e^-_\uparrow \mu^-_\downarrow \to e^-_\uparrow \mu^-_\downarrow \\ e^-_\downarrow \mu^-_\uparrow \to e^-_\downarrow \mu^-_\uparrow \end{array}\right\} J_z \pm 1$$

Angular momentum conservation and "helicity conservation" → backward scattering strongly suppressed.

5.3 The process $e^-e^+ \to \mu^-\mu^+$



Amplitude:

$$-i\mathcal{M} = \underbrace{(ie\overline{v}(p)\gamma^{\mu}u(k))}_{\text{electron current}}\underbrace{\left(-i\frac{g_{\mu\nu}}{q^2}\right)}_{\text{photon prop.}}\underbrace{(ie\overline{u}(k')\gamma^{\mu}v(p'))}_{\text{muon current}}$$

$$= ie^2\overline{v}(p)\gamma^{\mu}u(k)\frac{1}{q^2}\overline{u}(k')\gamma_{\mu}v(p')$$

As before: $|\mathcal{M}|^2 \to \overline{|\mathcal{M}|^2}$. Another approach: Crossing of $e^-\mu^- \to e^-\mu^-$ process: $\Rightarrow$ Exchange of $s \leftrightarrow t$:

$$\Rightarrow \overline{|\mathcal{M}|^2} = 2e^4\frac{t^2+u^2}{s^2}$$

in high-energy limit. Cross section in the CMS:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}\left(1+(\cos(\theta))^2\right)$$

Qualitative explanation: Remember that QED "conserves" helicity of fermion currents.

Remark. e^-/e^+ and μ^-/μ^+ couple with opposite helicity.

Contributions that conserve helicity:

1. Same helicity of e^- and μ^-
   Angular momentum conservation: Muon favors forward direction: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim (1+\cos(\theta))^2$
2. Opposite helicity of e^- and μ^-
   Muon favors backward direction: $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \sim (1-\cos(\theta)))^2$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\uparrow\uparrow}}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega_{\uparrow\downarrow}} = (1+\cos(\theta))^2 + (1-\cos(\theta))^2 = 2(1+(\cos(\theta))^2)$$

Total cross-section: Integrate over all angles:

$$\sigma(e^-e^+ \to \mu^+\mu^-) = \mathrm{d}_{\Omega}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \int_0^{2\pi}\mathrm{d}\phi\int_{-1}^{1}\mathrm{d}\cos\theta\frac{\alpha^2}{4s}(1+(\cos\theta)^2)$$

$$\sigma(e^-e^+ \to \mu^+\mu^-) = \frac{4}{3}\pi\frac{\alpha^2}{s}$$

The total cross-section which only depends on CM energy and on coupling strength α. From the TASSO experiment, a remarkable result: First principles $\to$ Prediction better than 1%. Corrections:Z^0 for $\sqrt{s} \geq m_Z = 91\,\mathrm{GeV}$ and weak interaction couples only to left-handed particles. Thus, inlt some processes contribute (need e_L). Also shift in $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

Cross-section for $e^+e^- \to q\bar{q}$ (jet production)

$$\sigma(e^-e^+ \to q\bar{q}) = \frac{4\pi}{3}\frac{\alpha^2}{s}\sum_i Q_i \cdot 3$$

where the sum is over all $q\bar{q}$ final states that can be produced at a given $\sqrt{s}$.
$\to$ Test quark charges, masses, color

$$R = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

5.4 Chirality and "helicity conservation"

Have seen: Convenient to express amplitudes with spinors that are eigenstates of the helicity operator $\hat{h}$. Remember: spinors we write down earlier were for special case in which spin and momentum are aligned $\to$ of limited use. More generally: Spinors that are helicity states are in polar coordinates

$$c \equiv \cos\left(\frac{\theta}{2}\right) \qquad s \equiv sin\left(\frac{\theta}{2}\right)$$

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ s_\phi, \phi \\ \frac{p}{E+m}\cdot c \\ \frac{p}{E+m}\cdot se^{i\phi} \end{pmatrix}, \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix}$$

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}\cdot s \\ -\frac{p}{E+m}\cdot ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}, \quad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

$$\text{with } \vec{p} = \begin{pmatrix} p\cos\theta\sin\phi \\ p\sin\theta\sin\phi \\ p\cos\theta \end{pmatrix}$$

$$\hat{h}v_\uparrow = +\frac{1}{2}u_\uparrow \ , \ \hat{h}u_\downarrow = -\frac{1}{2}u_\downarrow \text{ etc.}$$

Explicitly calculate $e^-_\uparrow e^+_\downarrow \to \mu^-_\uparrow \mu^+_\downarrow$.

- Electron along z-direction ($\theta = \phi = 0$)
- High energy approximation: $E \gg m$

1. $u_\uparrow(k) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$
2. $v_\downarrow(p) = \sqrt{E}\begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$

3. $\theta = \phi = 0$, $u_\uparrow(k') = \sqrt{E}\left(\begin{smallmatrix} c \\ s \\ c \\ s \end{smallmatrix}\right)$

4. Annihilation $(\theta = \pi - \theta, \phi = \pi)$, $v_\downarrow(p')\sqrt{E}\left(\begin{smallmatrix} s \\ -c \\ s \\ -c \end{smallmatrix}\right)$

Electron current:

$$j^\mu(e^-) = ie\bar{v}_\downarrow(p)\gamma^\mu u_\uparrow(k)$$

Calculate $j^\mu(e^-)$ component-wise:

$$\bar{v}_\downarrow\gamma^0 u_\uparrow(k) = v_\downarrow^+(p)\underbrace{\gamma^0\gamma^0}_{=I}u_\uparrow(k) = v_\downarrow^+ u_\uparrow(k) = v_1^* u_1 + v_2^* u_2 + v_3^* u_3 v_4^* u_4 = 0$$

Similarly:

$$\bar{v}_\downarrow\gamma^1 u_\uparrow(k) = v_\downarrow^+\gamma^0\gamma^1 u_\uparrow(k) = v_1^* u_4 + v_2^* u_3 + v_3^* u_2 + v_4^* u_1 = 0 - E - E + 0 = -2E$$

$$\bar{v}_\downarrow(p)\gamma^2 u_\uparrow(k) = -2iE$$

$$\bar{v}_\downarrow(p)\gamma^3 u_\uparrow(k) = 0$$

$$\Rightarrow j^\mu(e^-) = ie\begin{pmatrix} 0 \\ -2E \\ -2iE \\ 0 \end{pmatrix}$$

Muon current:

$$j^\mu(\mu^-) = ie\begin{pmatrix} 0 \\ -2E\cos\theta \\ 2iE \\ 2E\sin\theta \end{pmatrix}$$

Invariant amplitude:

$$-i\mathcal{M} = j^\mu(e^-)\frac{-i}{q^2}j_\mu(\mu^-) = ie\begin{pmatrix} 0 \\ -2E \\ -2iE \\ 0 \end{pmatrix}\frac{i}{q^2}\begin{pmatrix} 0 \\ -2E\cos\theta \\ 2iE \\ 2E\sin\theta \end{pmatrix} = \frac{ie^2}{q^2}4E^2\left(-1-\cos\theta\right) = -ie^2\left(1+\cos\theta\right)$$

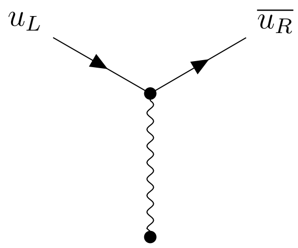
Similar calculation for the other (16 - 1) amplitudes.

→ 12 amplitudes are 0 (those that are not conserving the helicity of the currents). Only 4 survive. Fundamental reason: Chrial structure of QED. What does that mean?

Any QED current can be written in terms of left-handed and right-handed components.

$$\bar{u}\gamma^\mu = (\overline{u_L} + \overline{u_R})\gamma^\mu(u_L + u_R) = \overline{u_L}\gamma^\mu u_L + \overline{u_R}\gamma^\mu u_R$$

No mixing between left- and right-handed states.



is forbbiden by QED. → QED conserves the chirality of particles. e.g. for a particle-anti-particle coupling:

$$\bar{v}\gamma^\mu u = \overline{u_L}\gamma^\mu u_R + \overline{v_R}\gamma^\mu u_L$$

u_L, u_R, v_L, v_R are eigenstates of chirlaity operator

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{O}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & \mathbf{0}_2 \end{pmatrix}$$

Properties:

$$(\gamma^5)^2 = I \ ; \ (\gamma^5)^\dagger = \gamma^5 \ ; \ \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

Eigenvalues:

$$\begin{aligned} \gamma^5 u_L &= -u_L \\ \gamma^5 u_R &= +u_R \\ \gamma^5 v_R &= -v_R \\ \gamma^5 v_L &= +v_L \end{aligned}$$

Eigenstates can be projected onto each other by chirality projetion operator

$$P_L = \frac{1}{2}\left(1 - \gamma^5\right) \quad P_R = \frac{1}{2}\left(1 + \gamma^5\right)$$

$$\left.\begin{aligned} \Rightarrow u_L &= P_L u \\ u_R &= P_R u \end{aligned}\right\}\text{for particles}$$

$$\left.\begin{aligned} v_L &= P_R v \\ v_R &= P_L v \end{aligned}\right\}\text{for anti-particles}$$

In the high-energy limit helicity states $u_\uparrow, u_\downarrow, v_\uparrow, v_\downarrow$ are also eigenstates of γ^5, i.e

$$\begin{aligned} \gamma^5 u_\uparrow &= +u_\uparrow \ : \ u_\uparrow \to u_R \\ \gamma^5 u_\downarrow &= -u_\downarrow \ : \ u_\downarrow \to u_L \\ \gamma^5 v_\uparrow &= -v_\uparrow \ : \ v_\uparrow \to v_R \\ \gamma^5 v_\downarrow &= +v_\downarrow \ : \ v_\downarrow \to v_L \end{aligned}$$

At high energies chirality and helicity are identical. QED conserves chirality (at any energy) and helicity at high energies.

6 Weak interactions

Examples:

- β-decay: $n \to p + e^- + \overline{\nu_e} \quad \tau_{\text{free}} = 920\,\text{s}$
- $p \to n + e^+ + \nu_e \quad \tau_{\text{free}} > 10 \times 10^{30}$ yr, happens for bounded protons
- pio-decay:
  - $\pi^- \to \mu^- + \overline{\nu_\mu}, \tau = 2.6 \times 10^{-9}\,\text{s}$
  - $\pi^+ \to \mu^+ + \nu_\mu$
- muon-decay:
  - $\mu^- \to e^- + \nu_\mu + \overline{\nu_e}$
  - $\mu^+ \to e^+ + \overline{\nu_\mu} + \nu_e$

Remark. Not electromagnetically $\mu^- \to e^- + \gamma$, lepton number conservation

Decay of hadrons with strange, charm quarks:

$$K^+ \to \pi^+ \pi^0$$

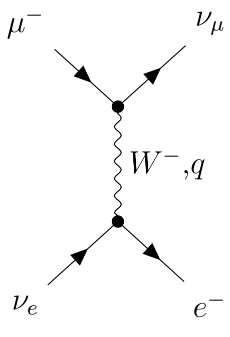
- Phenomenologically rather long-lived decays with a possible change of quark flavor
- Often "hidden" by electromagnetic or hadronic decays that happen much faster, $\pi^\pm, \mu^\pm$ are special since they can only decay via weak interaction

6.1 Charged weak interactions

Modern view: Exchange of W^+ or W^- boson.

Example 6.1.

$\mu^- \nu_e$ scattering



Fermion currents: Charge raising current:

$$J^\mu(\mu) = -i \frac{g}{\sqrt{2}} \overline{u}(\nu_\mu) \gamma^\mu \frac{1}{2}(1 - \gamma^5) u(\mu^-)$$

Charge lowering current:

$$J^\mu(e^-) = -i \underbrace{\frac{g}{\sqrt{2}}}_{\text{coupling constant}} \overline{u}(e^-) \underbrace{\gamma^\mu \frac{1}{2}(1-\gamma^5)}_{\text{structure of the interaction, "V-A"}} u(\nu_e)$$

"V-A": Vector-axialvector structure

Detour: Classify quantities according to their parity inversion properties. For a scalar:

$$x \xrightarrow{\hat{P}} x$$

A vector:

$$\vec{x} \overset{\hat{P}}{-} \vec{x}$$

Axial vector (cross-product):

$$\vec{L} \xrightarrow{\hat{P}} \vec{L}$$

Look back at QED currents:

$$j^\mu = \overline{u}\gamma^\mu u$$

under parity transformation?

1. Spinor:

$$u \xrightarrow{\hat{P}} \hat{P}u = \gamma^0 u$$

2. Adjoint spinor:

$$\overline{u} = u^\dagger \gamma^0 \to \overset{\hat{P}}{\gamma^0} u^\dagger \ \gamma^0 = u^\dagger \gamma^0 \gamma^0 = \overline{u}\gamma^0$$

$$\Rightarrow j^\mu = \overline{u}\gamma^\mu u \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^\mu\gamma^0 u$$

Components:

1. Time-like component:

$$\hat{P}j^0 = \overline{u}\underbrace{\gamma^0\gamma^0}_{=I}\gamma^0 u = \overline{u}\gamma^0 u = j^0$$

2. Space-like component:

$$\hat{P}j^k = \overline{u}\gamma^0\gamma^k\gamma^0 u = -\overline{u}\gamma^k\gamma^0\gamma^0 u = -\overline{u}\gamma^k u = -j^k$$

Parity changes sign of space-like components and keeps sign of time-like component.

$$\rightarrow j^\mu = \overline{u}\gamma^\mu u$$

is a vector-like quantity. $\rightarrow$ "V" in "V - A".

Remark.

$$\mathcal{M} = j^\mu \gamma_\mu = j^0 j_0 - j^k j_k \xrightarrow{\hat{P}} j^0 j_0 - (-j^k)(-j_k) = j^\mu j_\mu$$

Look at weak interaction currents ("A-term"):

$$j^\mu = \overline{u}\gamma^\mu\gamma^5 u \xrightarrow{\hat{P}} \overline{u}\gamma^0\gamma^5\gamma^0 u = -\overline{u}\gamma^0\gamma^\mu\gamma^0\gamma^5 u$$

1. Time-like component:

$$\hat{P}j^0 = -\overline{u}\gamma^0\gamma^0\gamma^0\gamma^5 u = -\overline{u}\gamma^0\gamma^5 u = -j^0$$

2. Space-like components:

$$\hat{P}j^k = -\overline{u}\gamma^0\gamma^k\gamma^0\gamma^5 u = +\overline{u}\gamma^k\gamma^5 u = \gamma^k$$

Also here: $j^\mu j_\mu$ is invariant under parity transformation. BUT: Scalar product

$$j_V^\mu \cdot j_A^\mu$$

$$j_V^0 \rightarrow j_V^0 \quad j_V^k \rightarrow -j_V^k$$

$$j_A^0 \rightarrow -j_A^0 \quad j_A^k \rightarrow j_A^k$$

$$j_V \cdot j_A \xrightarrow{\hat{P}} -j_V^0 j_A^0 + j_V^k j_A^k = -j_V j_A$$

$\rightarrow$ Parity violation.

V-A "vector-axialvector"-structure:

$$\overline{u}\gamma^\mu(1-\gamma^5)u = \underbrace{\overline{u}\gamma^\mu u}_{\text{vector current}} - \underbrace{\overline{u}\gamma^\mu\gamma^5 u}_{\text{axialvector current}}$$

Amplitude:

$$-i\mathcal{M} \sim j^\mu j'_\mu = (j_V - j_A)^\mu (j'_V - j'_A)_\mu$$

$$= \underbrace{j_V j'_V + j_A j'_A}_{\text{both scalar under parity}} - \underbrace{j_V j'_A - j_A j'_V}_{\text{both pseudoscalar under parity}}$$

$\rightarrow$ The amplitude does not behave well under space inversion $\rightarrow$ Parity violation.

First observed in Wu experiment (1957). Modern interpretation: Charged weak interaction couples exclusively to left-handed particles (charility -1) and right-handed anti-particles (chirality +1). Indeed from the charged weak current:

$$\overline{u}\gamma^\mu \frac{1}{2}(1-\gamma^5)u = \overline{u}\gamma^5 u_L = \overline{u_L}\gamma^\mu u_L$$

Chiral decomposition

Similarly for anti-particles:

$$\overline{v}\gamma^\mu \frac{1}{2}(1-\gamma^5)v = \overline{v}\gamma^\mu v_R = \overline{v_R}\gamma^\mu v_R$$

6.2 W Propagator and Fermi coupling constant

Exchange particle: W-boson (spin 1) like QED photon, charge $\pm$ 1 e and mass $m_w \sim$ 80 GeV which is very unlike a photon.

$$\frac{-ig_{\mu\nu}}{q^2} \rightarrow \frac{-i\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_W^2}\right)}{q^2 - m_W^2}$$

full W-boson propagator ($m_W \neq 0$). For many applications (e.g. decays) $\frac{q^2}{m_W^2} \ll 1$, so propagator becomes

$$\frac{ig_{\mu\nu}}{m_W^2}$$

effective W-propagator in the low-energy limit. → independent of momentum transfer. Amplitude of $\mu^- v_e$ scattering:

$$\begin{aligned} -i\mathcal{M} &= -i\frac{q}{\sqrt{2}}\overline{u}(\nu_\mu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(\mu^-)\left(\frac{ig_{\mu\nu}}{m_W^2}\right) \\ &= -i\frac{g}{\sqrt{2}}\overline{u}(e^-)\gamma^\nu\frac{1}{2}(1-\gamma^5)u(\nu_e) \\ &= -i\frac{q}{8m_W^2}\left(\overline{u}(\nu_\mu)\gamma^\mu(1-\gamma^5)u(\mu^-)\right)\left(\overline{u}(e^-)\gamma_\mu(1-\gamma^5)u(\nu_e)\right) \end{aligned}$$

no propagator (independence of g). "Effective four-fermion-interaction" (contact interaction) as long as $q^2 \ll m_W^2$ with effective coupling strength

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

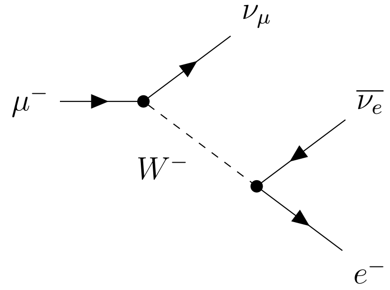
G_F: Fermi coupling constant. From muon-decay: $G_F = 1.166 \times 10^{-5}\,1/\mathrm{GeV}$. Approximation by contact interaction is valid at least for all weak decays since $Q \approx 1\,\mathrm{MeV} \ll$ 80 GeV.

6.3 Muon decay

Important weak decay to test properties of weak interactions.

$$\mu^- \rightarrow e^- + \overline{\nu_e} + \nu_\mu$$

Diagram:



$$q^2 < m_\mu^2 \ll m_W^2$$

→ contact approximation can be used. Amplitude:

$$-i\mathcal{M} = -i\frac{G_\pi}{\sqrt{2}}\left[\overline{u}(k)\gamma^\mu(1-\gamma^5)u(p)\right]\left[\overline{u}(p')\gamma^\mu(1-\gamma^5)v(k')\right]$$
$$= -i\frac{4G_F}{\sqrt{2}}\left(\overline{u}_L(k)\gamma^\mu u_L(p)\right)\left(\overline{u}_L(p')\gamma_\mu v_R(k')\right)$$

Helicity averaged amplitude:

$$\overline{|\mathcal{M}|}^2 = \frac{1}{2}\sum_{\text{final state helicities}} \overline{|\mathcal{M}|^2} = 64G_F^2(k\cdot p')(k'\cdot p)$$

Decay rate:

$$\mathrm{d}\Gamma\frac{1}{2E_A}\overline{|\mathcal{M}|^2}$$
$$= \frac{1}{2m_\mu}\overline{|\mathcal{M}|^2}\cdot\frac{\mathrm{d}^3p'}{(2\pi)^3\frac{3}{2}E'}\cdot\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega}\cdot\frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega'}$$

- E': Electron energy
- ω: ν_μ energy
- ω': $\overline{\nu_e}$ energy

Integrate over ν_μ and $\overline{\nu_e}$ phase space factor (since not observed in the experiment).

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E'} = \frac{G_F^2 m_\mu^2}{12\pi^3}\cdot E'^2\left(2-\frac{4E'}{m_\mu}\right)$$

Differential muon decay rate (muon rest-frame) as a function of electron energy E'. Possible corrections:

- Include electron mass
- W-propagator
- QED corrections

⇒ Very important validation of V-A structure of the weak interaction. Close to the kinematic boundry the electron is emitted in opposite direction w.r.t neutrino (momentum conservation). In this way the left-handed coupling (parity-violation) can be tested in this regime. Helicity configuration in this regime:

How to get polarized muons: Shoot pions on target (stop them):

$$\pi^+ \to \mu^+ + \nu_\mu$$

ν_μ is only left-handed. Spin of μ^- must be in opposite direction to direction of motion. Spin of the electron has to be in the direction of the spin of the muon (angular momentum conservation).

$\Rightarrow$ for θ: Angle between the spin of the muon and direction of motion of the electron. If left-handed coupling holds $\rightarrow \theta = 0 \Rightarrow$ should be suppressed.

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\Omega} \sim (1 - \cos\theta)$$

Matches well with experimental results. Total decay rate:

$$\Gamma(\mu \rightarrow e^- \overline{\nu_e} \nu_\mu) = \int \frac{\mathrm{d}\Gamma}{\mathrm{d}E'} \,\mathrm{d}E' = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

using $\Gamma = \frac{1}{\tau}$ and measurement of muon liftime $\tau = 2.2 \times 10^{-6}\,\mathrm{s}$ can determine G_F and the dimensionless coupling constant α_W

$$\alpha_W = \frac{g_W^2}{4\pi} \simeq \frac{1}{30}$$

Compare to QED: $\alpha \simeq \frac{1}{137} \simeq \alpha_W$. $\rightarrow$ In terms of coupling constants the electromagnetic interaction is weaker than weak interaction. Reminder: Weak interaction is weak at small energies (taht we have in decays) because $\frac{1}{m_W^2}$ dominates boson propagator. Total decay rate is also an excellent test of lepton universality. Expect:

$$\Gamma(\tau^- \rightarrow e^- \overline{\nu_e} \nu_\tau) = \Gamma(\mu^- \rightarrow e^- \overline{\nu_e} \nu_\mu) \left(\frac{m_\tau}{m_\mu}\right)^5$$

$\rightarrow$ Excellent agreement with experiment. Consquence: W-boson couples with equal strength $\frac{g_W}{\sqrt{2}}$ to all lepton doublets.

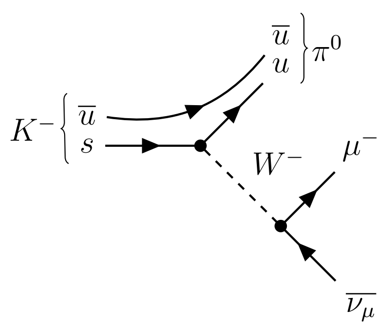
$$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L \begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L \begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L \begin{pmatrix}e^+ \\ \nu_e\end{pmatrix}_R \begin{pmatrix}\mu^+ \\ \nu_\mu\end{pmatrix}_R \begin{pmatrix}\tau^+ \\ \nu_\tau\end{pmatrix}_R$$

6.4 Coupling of the W to quarks

For quarks inter-generation coupling is observed (as opposed to the lepton sector).

Example 6.2.

$$K^- \rightarrow \pi^0 \mu^- \overline{\nu_\mu}$$

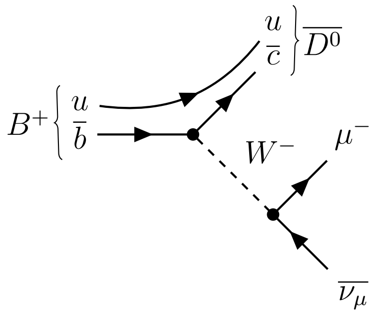


$\Rightarrow s$ couples to a u-quark. The quark generations:

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L$$

Example 6.3.

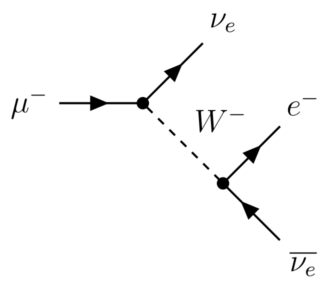
$$B^+ \to \overline{D^0}\mu^+\nu_\mu :$$



Not observed in the lepton sector.

Example 6.4.

$$\mu^- \to \nu_e e^- \overline{\nu_e}$$



NO!

Solution: (Two generations of quarks) to restore unified coupling:

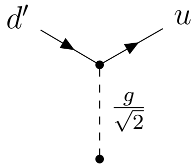
$$\underbrace{\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix}}_{\text{mass eigenstates (flavor eigenstates)}} \quad \to \quad \underbrace{\begin{pmatrix} u \\ d' \end{pmatrix} \begin{pmatrix} c \\ s' \end{pmatrix}}_{\text{weak eigenstates}}$$

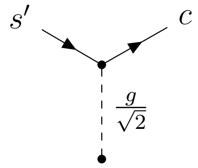
with

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}$$

with $\sin\theta_c \simeq 0.22 \to \theta_c \simeq 127°$, the Cabbibo angel.

d': Superposition of the down-type quarks. For these weak eigenstates only intra generation coupling (like leptons) with universal coupling is possible.

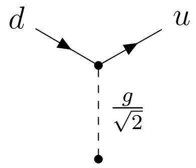




$$\frac{g}{\sqrt{2}}\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma^5)d' \qquad \frac{g}{\sqrt{2}}\bar{c}\gamma^\mu\frac{1}{2}(1-\gamma^5)s'$$

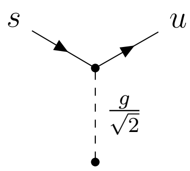
$\rightarrow$ Must properly rotate the mass-eigenstates when using quarks in weak interactions.

Example 6.5. $d \rightarrow u$-transition: Only d' weak eigenstate involved.



$$\frac{g}{\sqrt{2}}\cos\theta_c \simeq 97.5\,\%\frac{g}{\sqrt{2}}$$

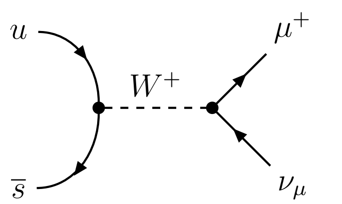
Example 6.6. $s \rightarrow u$-transition:



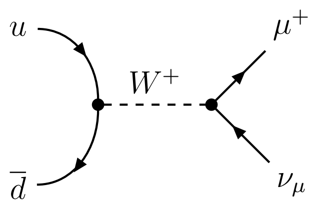
$$\frac{g}{\sqrt{2}}\cos\theta_c \simeq 22\,\%\frac{g}{\sqrt{2}}$$

$\Rightarrow s \rightarrow u$ suppressed w.r.t. $d \rightarrow u$ by $\left(\frac{0.22}{0{,}975}\right)^2$ in the decay rate.

Experimental test: $K^+ \rightarrow \mu^+\nu_\mu$ versus $\pi^+ \rightarrow \mu^+\nu_\mu$



$\hat{=}$



$$\frac{g}{\sqrt{2}}(\sin\theta_c)^* \text{ and } \frac{g}{\sqrt{2}} \qquad \frac{g}{\sqrt{2}}(\sin\theta_w)^* \text{ and } \frac{g}{\sqrt{2}}$$

$$J^\mu(us) = -\frac{ig}{\sqrt{2}}\overline{d'}\gamma^\mu\frac{1}{2}(1-\gamma^5)u$$

$$
\begin{aligned}
&= \frac{-ig}{\sqrt{2}}(\sin\theta_c{}^*)\overline{s}\gamma^\mu \frac{1}{2}(1-\gamma^5)u \\
J^\mu(ud) &= \frac{-ig}{\sqrt{2}}(\cos\theta_c)^*\overline{d}\gamma^\mu \frac{1}{2}(1-\gamma^5)u \\
\frac{\Gamma(K^+ \to \mu^+\nu_\mu)}{\pi^+ \to \mu^+\nu_\mu} &= \frac{(\sin\theta_c)^2}{(\cos\theta_c)^2} \simeq 5\,\%
\end{aligned}
$$

Experimentally: $\frac{\Gamma(K^+)}{\Gamma(\pi^+)} = 130\,\%$. Why? $m(k^+ \gg m(\pi^+))$: Much more phase space to decay into. After phase space (and hadronic) corrections very good agreement with predictions.

6.5 The CKM-matrix

Have three quark generations:

$$
\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}
$$

Where V_{CKM} is a unitary rotation matrix.
Unitarity:

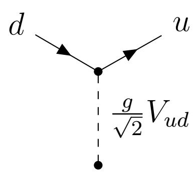
- 3 rotation angles $\theta_{12}, \theta_{13}, \theta_{23}$
- 1 complex phase (CP violation phase) δ_{13}

$$
V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cd} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & c_{13} & \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

with $c_{ij} = \cos\theta_{ij}$ and $j = \sin\theta_{ij}$. Structure of current:

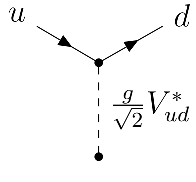
$$
J^\mu = -\frac{ig}{\sqrt{2}}(\overline{uct})\gamma^\mu \frac{1}{2}(1-\gamma^5) V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}
$$

Example 6.7.



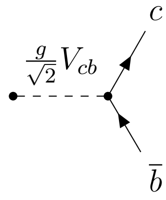
$$
J^\mu = -\frac{ig}{\sqrt{2}}\overline{u}\gamma^\mu \frac{1}{2}(1-\gamma^5)V_{ud}d
$$

Example 6.8.



$$
J^\mu = -\frac{ig}{\sqrt{2}}\overline{d}V_{ud}^*\gamma^\mu \frac{1}{2}(1-\gamma^5)u
$$

Example 6.9.



$$J^\mu = -\frac{g}{\sqrt{2}}\bar{c}\gamma^\mu \frac{1}{2}(1-\gamma^5)V_{cb}b$$

Experimental results on the absolute values of the elements of V_{CKM}:

$$|V_{CKM}| = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

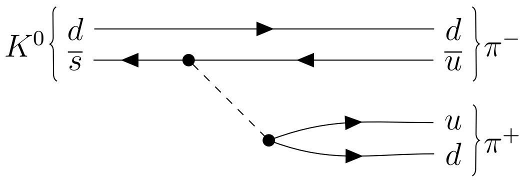
$\rightarrow$ Intra-generation-coupling are strongest. Rotation between flavor (mass) and weak eigenstates is small ($\theta_{12} = 13°, \theta_{23} = 2.3°, \theta_{13} = 0.2°$)

6.6 Strangeness oscillations

Consider a system of neutral K mesons which are the lightest mesons containing s-quarks.

$$K^0 = (d\bar{s})\ ,\ \overline{K^0}(\bar{d}, s)$$

Flavor eigenstates produced in strong interaction, e.g. $\pi^- + p \rightarrow \Lambda + K^0$ or $\overline{p} + p \rightarrow K^+ + \overline{K^0} + \pi^-$. Both, K^0 and $\overline{K^0}$, decay into $\pi\pi$ and $\pi\pi\pi$, e.g.



Experimentally: Observe two different kaons with vastly different lifetimes.

$$K_S^0 \rightarrow \pi\pi\ :\quad \tau(K_s^0) = 9\times 10^{-11}\,\text{s}$$
$$K_L^0 \rightarrow \pi\pi\pi\ :\quad \tau(K_L^0) = 5\times 10^{-8}\,\text{s}$$

Liftime difference: Phase space ($\pi\pi$ more phase). K^0S, K_L^0 are decaying weak eigenstates with well-defined lifetimes and masses (mass eigenstates of propagating particles).

Proposal: K_S^0 and K_L^0 are different mixtures of K^0 and $\overline{K^0}$. Flavor content of production: Both K_S^0 and K_L^0: 50 %K^0 and $\overline{K^0}$ each.

$$K_S^0(t=0) = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\overline{K^0}\right\rangle\right) K_L^0(t=0) = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\overline{K^0}\right\rangle\right)$$

Attention: This only works if <u>no</u> CP violation! Propagate in rest-frame:

$$\left|K_S^0(t)\right. = \left|K_S^0(t=0)\right\rangle \cdot \underbrace{\exp\left\{-im_S + \frac{S}{2}\cdot t\right\}}_{\text{Exponential decay}}$$

$$\left|K_L^0(t) = \left|K_L^0(t=0)\right\rangle \cdot \exp\left\{-im_L + \frac{L}{2}\cdot t\right\}$$

K_S^0 and K_L^0 are particles that propagate and that decay. Probability to observe a $\overline{K^0}$ flavor state at time t when K^0 flavor state was reduced at time $t=0$?

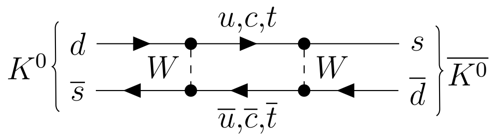
$$P(t) = \left|\left\langle \overline{K^0}(t) \middle| K^0(t) \right\rangle\right|^2$$

Express K^0 and $\overline{K^0}$ in terms of K_L^0 and K_S^0

$$\begin{aligned} \left|K^0\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|K_L^0\right\rangle + \left|K_S^0\right\rangle\right) \\ \left|\overline{K^0 i}\right\rangle &= \frac{1}{\sqrt{2}}\left(\left|K_L^0\right\rangle - \left|K_S^0\right\rangle\right) \end{aligned}$$

$$\begin{aligned} P(t) &= \left|\frac{1}{2}\left(\left\langle K_L^0(t)\right| - \left\langle K_S^0(t)\right|\right)\left(\left|K_L^0(0)\right\rangle + \left|K_S^0(t)\right\rangle\right)\right|^2 \\ \frac{1}{4}&\left|e^{im_L - \frac{\Gamma_L}{2}t}\left\langle K_L^0(0)\middle|k_L^0(0)\right\rangle - e^{im_S t - \frac{\Gamma_S}{2}t}\left\langle K_S^0(0)\middle|K_S^0(t)\right\rangle\right|^2 \\ &= \frac{1}{4}\left(\underbrace{e^{-\Gamma_L t} + e^{-\Gamma_S t}}_{\text{decay}} - \underbrace{2\cos\left(\Delta m t\right)e^{\frac{-\Gamma_L+\Gamma_S}{2}t}}_{\text{Oscillation } K^0 \leftrightarrow \overline{K^0}}\right) \end{aligned}$$

The strangeness oscillation. Mixing of K^0 and $\overline{K^0}$ will produce a $\overline{K^0}$ component. Fundamental process beind strangeness oscillations:



A higher order box diagram. Hwo to measure that?

1. Produce K^0 and $\overline{K^0}$ in $p\overline{p}$ collision in equal rate:

$$\begin{aligned} p\overline{p} &\rightarrow K^-\pi^+\mathrm{K}^0 \\ & p\overline{p}K^+\pi^-\overline{K^0} \end{aligned}$$

   The $K^\pm$ are used to tag the K^0 at production

2. Propagate $K^0/\overline{K^0}$ (i.e. K_S^0 and K_L^0) in the experiment before it decays. Propagation time $\sim$ distance production and decay

3. Use flavor sensitive decay modes to tag if K^0 or $\overline{K^0}$ decayed

$$\begin{cases} K^0 \to \pi^- e^+ \nu_e \\ \overline{K^0} \to \pi^+ e^- \overline{\nu_e} \end{cases} \quad \text{use lepton charge to tag}$$

4. Measure time-dependent lepton charge rate asymmetry

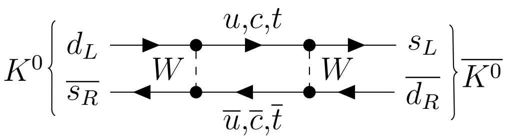
$$A(t) = \frac{(R_+ + \overline{R_-})(R_- + \overline{R_+})}{(R_+ + \overline{R_-}) + (R_- + \overline{R_+})}$$

R_+: Rate of K^0 decays (positron track) with K^0 at production time
$\overline{R_-}$: Rate of $\overline{K^0}$ decays (electron tracks) with $\overline{K^0}$ at production time

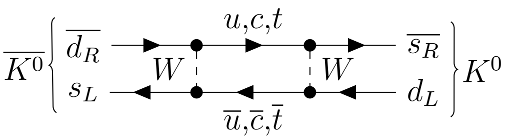
Oscillation pattern is observed. Used to extract $\Delta m = m_L - m_S = 348 \times 10^{-6}\,\text{eV}$. Small, but not zero $\to$ oscillations.

6.7 CP violation in the Kaon system

Weak interactions violate parity (maximally), because u_L couples to W, $\hat{P}u_L = u_R$ does not and charge conjugation u_L to W, $\hat{C}u_L = v_L$ does not. Is CP violated? u_L couples to W, $\hat{C}\hat{P}u_L = v_R$ does too! $\to$ weak interaction should conserve the product of C and P! (Assumed to be) true for the lepton sector. Quark sector: Complex phase of CKM matrix enables CP violation. $\overline{K^0} - K^0$ oscillations:



CP mirrored process:



CP invariance of weak interactions equal for $K^0 \to \overline{K^0}$ and $\overline{K^0} \to K^0$ $\to$ onlt possible if CKM elements are real. If $\delta_{13} \neq 0$ weak interation violates CP.
Test: Are weak eigenstates K^0_S and K^0_L CP eigenstates? CP eigenstates:

- Are K^0 and $\overline{K^0}$ CP eigenstates?

$$P = (-1)^k \quad K^0, \overline{K^0}\ ,\ L = 0\ ,\ \Rightarrow P = -1$$
$$\hat{P}\left|K^0\right\rangle = -\left|K^0\right\rangle \quad \hat{C}\left|K^0\right\rangle = \hat{C}\left|d\bar{s}\right\rangle = \left|\bar{d}s\right\rangle$$
$$\hat{P}\left|\overline{K^0}\right\rangle = -\left|\overline{K^0}\right\rangle$$

$$\Rightarrow CP(K^0) = \left|\overline{K^0}\right\rangle$$
$$CP(\overline{K^0}) = \left|K^0\right\rangle$$

$\Rightarrow |K^0\rangle$ and $\left|\overline{K^0}\right\rangle$ are not CP eigenstates

- Rather straigth forward to show that

  $$\left|K_1^0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle + \left|\overline{K^0}\right\rangle\right) \quad CP = +1 \left|K_2^0\right\rangle = \frac{1}{\sqrt{2}}\left(\left|K^0\right\rangle - \left|\overline{K^0}\right\rangle\right) \quad CP = -1$$

  (These were K_S^0 and K_L^0 before, if CP is not violated) $|K_1^0\rangle$ and $|K_2^0\rangle$ have CP-conserving decays:

  $$\left|K_1^0\right\rangle \to \pi^+\pi^-$$
  $$\left|K_2^0\right\rangle \to \pi^+\pi^-\pi^0$$

  Since: $CP(\pi^+\pi^-) = +(\pi^+\pi^-)$ and $CP(\pi^+\pi^-\pi^0) = -(\pi^+\pi^-\pi^0)$.

  $$\hat{P}(\pi^+\pi^-) = (-1)^2 P(\pi^+)P(\pi^-) = (-1)^0(-1)(-1) = +1(\pi^+\pi^-)$$
  $$\hat{P}(\pi^+\pi^-\pi^0) = -1(\pi^+\pi^-\pi^0)$$
  $$\hat{C}(\pi^+\pi^-) = C(\overline{u}d\overline{d}u) = u\overline{d}du = +1(\pi^-\pi^+) = +1(pi^+\pi^-)$$
  $$\Rightarrow CP(\pi^+\pi^-) = +(\pi^+\pi^-)$$

- $|K_1^0\rangle$ and $|K_2^0\rangle$ are CP eigenstates. $|K_S^0\rangle$ and $|K_R^0\rangle$ decay into CP eigenstates. $\to$ in absence of CP violation

  $$\left|K_1^0\right\rangle = \left|K_S^0\right\rangle$$
  $$\left|K_2^0\right\rangle = \left|K_L^0\right\rangle$$

Remark. In the strangeness oscillation we made this equality.

How to test the equality experimentally? We know the $|K_S^0\rangle$ component (i.e. the $|K_1^0\rangle$ component) dies quickly. $\to$ for $t \gg \tau_1 = \tau_S(10^{-11}\,\text{s})$ expect to see only $K_L^0 \to \pi^+\pi^-\pi^0$ decays, never into $\pi^+\pi^-$.

- CP violating decay $|K_L^0\rangle \to \pi\pi$
- CP conserving decay $|K_K^0\rangle \to \pi\pi\pi$

First detection of $\pi^+\pi^-$ decays of the $|K_L^0\rangle$ 1964 by Cronin and Fitch (NP 1980): 0.2 % of all $|K_L^0\rangle$ decays are into $\pi^+\pi^-$. $\to$ Weak interaction violates CP (at a very low leve!). Is consequence of that $K^0 \to \overline{K^0}$ and $\overline{K^0} \to K^0$ happen at slightly different rates.

- Are ingredient to explain matter-antimatter asymmetry in the universe

- Formally weak eigenstates are:

$$\left|K_S^0\right\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left[\left|K_1^-\right\rangle + \epsilon\left|K_2^0\right\rangle\right] = \frac{1}{2(1+|\epsilon|^2)}\left[(1+\epsilon)\left|K^0\right\rangle + (1-\epsilon)\left|\overline{K^0}\right\rangle\right]$$

with

$$BR(K_S^0 \to \pi\pi) \simeq 100\,\%$$
$$BR(K_S^0 \to \pi\pi\pi) \simeq 4\times 10^{-1}\,\%$$

$$\left|K_L^0\right\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left[\epsilon\left|K_1^0\right\rangle + \left|K_2^0\right\rangle\right]$$

with

$$BR(K_L^0 \to \pi\pi\pi) \simeq 32\,\%$$
$$BR(K_L^0 \to \pi\pi) \simeq 2\times 10^{-3}\,\%$$

(CP violating part)

$$|\epsilon| = 2.26\times 10^{-3}$$

- Indirect CP violation: $K^0 - \overline{K^0}$ mixing at different rates leads to CP violation iin the decay
- Direct CP violation: CP violated directly in the decay i.e. $K_2 \to \pi\pi$
$\Rightarrow$ Indirect CP violation is dominant factor!

6.8 Massive neutrino: Neutrino oscillations

In the SM, $m(\nu_e) = m_{\nu_\mu} = m_{\nu_\tau} = 0$. No deeper reason, besides experimental observations:

$$m(\nu_e) < 2\,\text{eV}(\ \beta\text{-decay spectrum})$$
$$\sum_{i=0}^{3} m_{\nu_i} \leq 1\,\text{eV}(\ \text{cosmological structure formation})$$

From observations of stellar neutrinos: Flux $(\nu_e) \simeq \frac{1}{2}$ expected flux from standard model of the sun. Reason:

$$\nu_e \to \nu_\mu \quad (\nu_\tau)$$

on their way to earth: Neutrino oscillation. At least one neutrino must have a mass > 0 (of strangeness oscillations).

Phenomenological description (two flavors):

$$\underbrace{\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix}}_{\text{weak flavor eigenstates}} = \begin{pmatrix}\cos\theta_{12} & \sin\theta_{12}\\ -\sin\theta_{12} & \cos\theta_{12}\end{pmatrix} \underbrace{\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}}_{\text{mass eigenstates}}$$

Very similar to Cabbibo for quark mixing. Time evolution of a $|\nu_e\rangle$ - state (propagation in x-direction)

$$|\nu_e\rangle = \underbrace{\cos\theta_{12}e^{-i(E_1 t - p_1 x)}\,|\nu_1(0)\rangle + \sin\theta_{12}e^{-iE_2 t - p_2 x}\,|\nu_2(0)\rangle}_{\text{Coherent interference of mass eigenstates}}$$

$$E_i = \left(p_i^2 + m_i^2\right)^{\frac{1}{2}} \simeq E_i\left(1 + \frac{m_i^2}{2E_i^2}\right)^{\frac{1}{2}} \simeq E_i\left(1 + \frac{m_i^2}{2E_i^2}\right)$$

Start with a pure ν_e state. Probability of recovering a ν_e after time t:

$$P(t) = \left|\langle\nu_e(t)|\nu_e(0)\rangle\right|^2 = \dots = 1 - (\sin(\theta_{12}))^2 \cdot \left(\sin\left(\frac{\Delta m_{21}^2}{4}\frac{L}{E}\right)\right)^2$$

with $E_1 = E_2 = E = p$ (small masses) and $L = c\cdot t$ (travelling distance) and $\Delta m_{21}^2 - m_1^2$ the difference of the squared masses. In useful units:

$$P(t) = 1 - (\sin(2\theta_{12}))^2 \cdot \left(\sin\left(\frac{1.27\Delta m_{21}^2[\text{eV}]}{E[\text{GeV}]}\right)\right)^2$$

Oscillation length

$$L_{\text{osc}} = \frac{4\pi E}{\Delta m_{21}^2} = \frac{\pi E[\text{GeV}]}{1.27 m_{21}^2[\text{eV}^2]}\cdot m$$

Typical values:

$$\left.\begin{aligned}&E = 1\,\text{MeV}\\ &\Delta m_{21}^2 = 8\times10^{-5}\,\text{eV}\end{aligned}\right\} \Rightarrow L_{\text{osc}} = 30\,\text{km}$$

$\rightarrow$ Quantum mechanical interference on very large scales. Three very important regimes:

1. $$\frac{\Delta m_{21}^2}{4}\cdot\frac{L}{E} \ll 1$$

   No oscillation yet, $\frac{L}{E}$ too small

2. $$\frac{\Delta m_{21}^2}{4}\cdot\frac{L}{E} \simeq 1$$

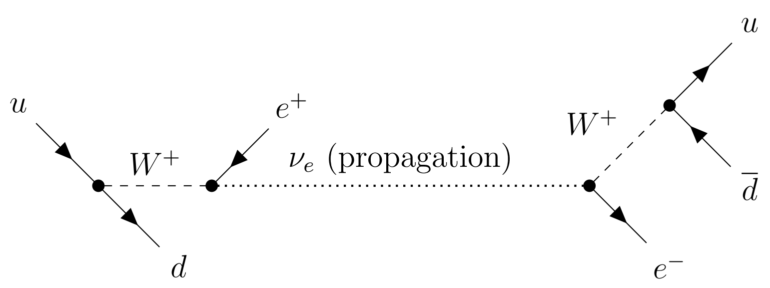
   maximum oscillations

3.

$$\frac{\Delta m_{21}^2}{4} \cdot \frac{L}{E} \gg 1$$

many oscillation lengths: $\rightarrow$ observe average oscillation amplitude: $\langle \left(\sin\left(\frac{\Delta m_{21}^2}{4}\frac{L}{E}\right)\right)^2 \rangle = \frac{1}{2}$

Typical feynman-diagram of propagation e.g. $\nu_e \rightarrow \nu_e$ (e.g. in the sun electron capture $p + e^- \rightarrow n + \nu_e$



Couplings: $\frac{g}{\sqrt{2}}\cos\theta_{12}$ for $\nu_e \rightarrow \nu_1$; $\frac{g}{\sqrt{2}}$ for $\nu_e \rightarrow \nu_e\nu_2$. For invariant amplitude:Must sum diagrams of ν_1 and ν_2 (like in the case of quark mixing).

6.9 3 generation oscillations

$$\underbrace{\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}}_{\text{weak flavor eigenstates}} = \underbrace{\begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}}_{\text{complex PMNS mixing matrix}} \underbrace{\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}}_{\text{mass eigenstates}}$$

The Pontecorvo-Maki-Nakagawa-Sakata-Matrix (similar to CKM-matrix).

Assume unitary: Three rotation angles $\theta_{12}, \theta_{13}, \theta_{23}$, one complex phase δ (CP violation in lepton sector if $\delta \neq 0$, not measured yet).

So e.g. $\nu_e \rightarrow \nu_\mu$ but also $\nu_e \rightarrow \nu_\tau \rightarrow$ Complex oscillation pattern. Strongly depend on:

$$\Delta^2 m_{21} = m_2^2 - m_1^2$$
$$\Delta^2 m_{31} = m_3^2 - m_1^2$$
$$\Delta^2 m_{32} = m_3^2 - m_2^2$$

and $\theta_{12}, \theta_{13}, \theta_{23} \rightarrow$ oscillation amplitudes.

Summary of exerpimental results:

$$\Delta m_{21}^2 = 7.6 \times 10^{-5}\,\text{eV}^2$$
$$\Delta m_{32}^2 = 2.4 \times 10^{-3}\,\text{eV}^2$$

$$\rightarrow \left|\Delta m_{31}^2\right| \simeq \left|\Delta m_{32}^2\right| \text{ since } \left|m_{32}^2\right| \gg \left|\Delta m_{21}^2\right|$$
$$\text{and } \Delta m_{21}^2 + \Delta m_{32}^2 - \Delta m_{31}^2 = 0$$

$$\theta_{12} = 33.6^{+0.8}_{-0.9} \text{ (solar neutrinos)}$$
$$\theta_{23} = 47.2^{+1.9}_{-3.9} \text{ (atmospheric neutrinos)}$$
$$\theta_{13} = 8.5^{+0.15}_{-0.15} \text{ (reactor neutrinos)}$$

$\Rightarrow$ Much more mixing in the neutrinos compared to quarks. PMNS matrix:

$$|U| = \begin{pmatrix} 0.82 & 0.54 & 0.15 \\ 0.35 & 0.70 & 0.62 \\ 0.44 & 0.45 & 0.77 \end{pmatrix}$$

CP violating phase δ not yet measured (although T2K in Japan: 2σ effect that oscillation of ν_μ into ν_e is larger than oscillation of $\overline{\nu_\mu}$ into $\overline{\nu_e}$).

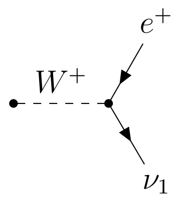
ν_e is dominated by ν_1 mass eigenstate, ν_τ is a mixture of ν_2 and ν_3 (and a bit of ν_1). General structure of weak lepton current:

$$J^\mu = -i\frac{g}{\sqrt{2}}(\overline{e\mu\tau})\gamma^\mu\frac{1}{2}(1-\gamma^5)U\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}$$

cf: Quark sector:

$$J^\mu = -i\frac{g}{\sqrt{2}}(\overline{uct})\gamma^\mu\frac{1}{2}U_{CKM}\begin{pmatrix}d\\ s\\ b\end{pmatrix}$$

Example 6.10. Example for couplings:

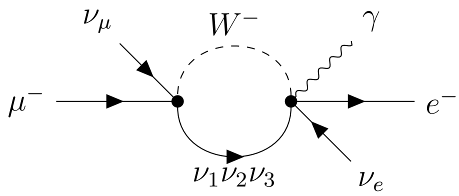


$$J^\mu = -\frac{ig}{\sqrt{2}}\overline{\nu_1}U_{e1}^*\frac{\gamma^\mu}{2}(1-\gamma^5)e$$

Interesting consquence of oscillations: Lepton-number violating decays are possible:

$$\mu^- \rightarrow e^- + \gamma$$

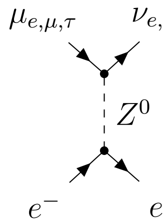
which violates both, L_e and L_μ. Diagram with oscillations allowed:



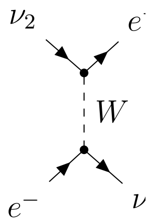
should be allowe. Current limit: $BR(\mu^- \to e^-\gamma) < 5 \times 10^{-13}$. Expectation from the standard model: $BR(\mu^- \to e^-\gamma \simeq 1 \times 10^{-55}$, but larger in some supersymmetric scenarios.

6.10 Oscillations in matter

Neutrino propagation is affected by the presence of matter.



The neutral current. **Only** for ν_e:



Optics: Scattering of photons on electron of matter. $\to$ Smaller velocity $v = \frac{s}{n}$ $\to$ effective photon mass. For neutrinos: Shift of the ν_e part of mass eigenstate that propagates with repect to ν_μ, ν_τ part. $\to$ smaller velocity of ν_e part $\to$ effective neutrino mass in matter.

- Change oscillation behaviour
- Can lead to resonant amplification of oscillation amplitudes (MSW effect)

Need to be taken into account for the sun or for propagation through earth.

6.11 Solar netrino measurements

- Nuclear fusion in the sun produces $\sim 2 \times 10^{38} \nu_e$ per second

- Spectrum from solar fusion cicle
- *pp* neutrinos: $p + p \to d + e^+ + \nu_e$: $E_{\max} = 0.42\,\mathrm{MeV}$
- *pep* neutrinos: $p + p + e^- \to d + \nu_e$: $E_{\max} = 1.44\,\mathrm{MeV}$
- *hep* neutrinos: $^3He + p \to ^4 He + \nu_e + e^-$: $E_{\max} = 18.8\,\mathrm{MeV}$
- ^7Be neutrinos: $^4He + ^3 He \to ^7 Be$: $E_{\max} = 0.384\,\mathrm{MeV}$
  $^7Be + e^- \to ^7 Li + \nu_e$: $E_{\max} = 0.862\,\mathrm{MeV}$
- ^8Be neutrinos: $^7Be + p \to ^8 B$: $E_{\max} = 14.1\,\mathrm{MeV}$
  $^8B \to ^8 Be + e^+ + \nu_e$

Measurements of ν_e-flux at earth ("dissappearence experiments").

- Pioneering: Homestake experiment: $615t$ of C_2Cl_4 (dry cleaning fluid). Detection:

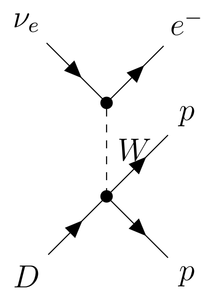
$$\nu_e + ^{37} Cl \to ^{37} Ar + e^- \to ^{37} Cl + \nu_2 + \gamma$$

$$t_{\frac{1}{2}} \simeq 35\,\mathrm{d}$$

  Energy threshold: $E_{\mathrm{th}} = 0.81\,\mathrm{MeV} \to$ mainly sensitive t ^8B neutrinos
  $\to$ Found 0.5 events per day. $\sim \frac{1}{3}$ of expectation from standard solar model.
  NP 2002 for discovery of neutrino oscillations.
  Others: SAGE, GALLEK

- Most modern measurement: SNO
  - Dissappearence *and* appearence measurement
  - Sensitive to ν_e, ν_μ, ν_τ
  - 1200 t of heavy water
  - 12 m diameter vessel
  - 9600 PMTs
  - Detection process:
    1. Charged-current process on nucleus:

$$\nu_e + d \to p + p + e^-$$

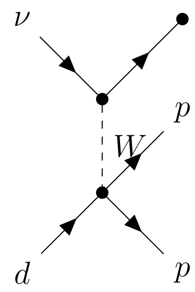


Why not for ν_μ, ν_τ? Energy threshold to produce μ^-, τ^0
$\Rightarrow$ CC on nucleus only sensitive to ν_e.

2. Neutral current process on nucleus:

$$\nu_{e,\mu,\tau} + d \to p + n + \nu_{e,\mu,\tau}$$
$$n + d \to He + \gamma$$

where γ does compton scattering on electrons.



NC on nucleus sensitive to ν_e, ν_μ, ν_τ

3. Elastic scattering off atomic electrons:

$$\nu_{e,\mu,\tau} + e^- \to \nu_{e,\mu,\tau} + e^-$$

Sensitive to $\Phi(\nu_e) + 0.15\,(\Phi(\nu_\mu + \Phi(\nu_\tau)))$

4. Results:
   * $\Phi(\nu_e) = 1.76 \times 10^{-6}/(\mathrm{cm}^2\,\mathrm{s})$
   * $\Phi(\nu_\tau) + \Phi(\nu_\tau) = 3.41 \times 10^{-6}/(\mathrm{cm}^2\,\mathrm{s})$
   * Total: $5.17 \times 10^{-6}/(\mathrm{cm}^2\,\mathrm{s})$
   * Prediction solar standard model: $\Phi(\nu_e+\nu_\mu+\nu_\tau) = 5.1 \times 10^{-6}/(\mathrm{cm}^2\,\mathrm{s})$
   * Result mixing angle: $\theta_{12} \simeq 34°$
   * $\Delta m^2_{21} = 7.6 \times 10^{-5}\,\mathrm{eV}^2$
   * NP 2015

6.12 Long-baseline experiments

Atmospheric neutrinos:

- CRs interacting in Earth's atmosphere
- $$CR \to PP \to XY \to \pi^+.\pi^-$$
  $$\pi^- \to \mu^-\nu_\mu$$
  $$\mu^- \to e^-\nu_\mu\overline{e}$$

  (similar for π^+).

$\rightarrow$ expect $\overset{(-)}{\nu}_\mu = 2\Phi(\overset{(-)}{\nu}_e)$. Typical energies: $0.1-10\,\text{GeV}$. Detection: e.g. Superkamiokande (50000 t of H_2O).

$$\left(\overset{(-)}{\nu}_\mu, \overset{(-)}{\nu}_e\right) + n \rightarrow \left(\mu^\pm, e^\pm\right) + p$$

Use Cherenkov light of the outgoing leptons for detection.

- Different Zenith angles θ probe a large range of path lengths
- Measurements:
  - Large deficit of up-going $\bar{\nu}_\mu$
  - No deficit of $\overset{(-)}{\nu}_e$

  $\Rightarrow$ Oscillation $\nu_\mu \rightarrow \nu_\tau$ in the Earth at these $\frac{L}{E}$ values
- No oscillations for ν_e

Results:

- $\theta_{23} \simeq 45°$
- $|\Delta m^2_{32}| \simeq 2.4 \times 10^{-3}\,\text{eV}^2$

Long-baseline experiments: Accelerator and detector on different locations on the Earth's surface ($d \sim 10^6\,\text{m}$).

- MINOS: ν_μ disappearance
- OPERA: ν_τ appearance (only 5 ν_τ in several years!)

$\rightarrow$ Confirmation of atmospheric result.

6.13 Reactor experiments

- Produced in β-decays of radioisotopes ($^{235}U, ^{238}U, ^{239}Pu$)

Consider full oscillation picture for $\overline{\nu_e}$ produced in a reactor

$$P(\overline{\nu_e} \rightarrow \overline{\nu_e}) = \underbrace{1 - 4c^2_{12}s^2_{12}c^2_{13}\left(\sin\left(\frac{\Delta m^2_{21}}{4}\frac{L}{E}\right)\right)^2}_{(*)} - \underbrace{4c^2_{13}s^2_{13}\left(\sin\left(\frac{\Delta m^2_{32}}{4}\frac{L}{E}\right)\right)}_{(**)}$$

$$c_{ij} = \cos\theta_{ij}\ ,\ \ s_{ij} = \sin\theta_{ij}$$

Since $|\Delta m^2_{21}| \ll |\Delta m^2_{32}|$ term $(*)$ oscillates slowly as $\frac{L}{E}$ increases, term $(**)$ oscillates faster. First minimum at $\frac{\Delta m^2_{32}}{4}\frac{L}{E} = \frac{\pi}{2}$. 2012: Daya Bay:

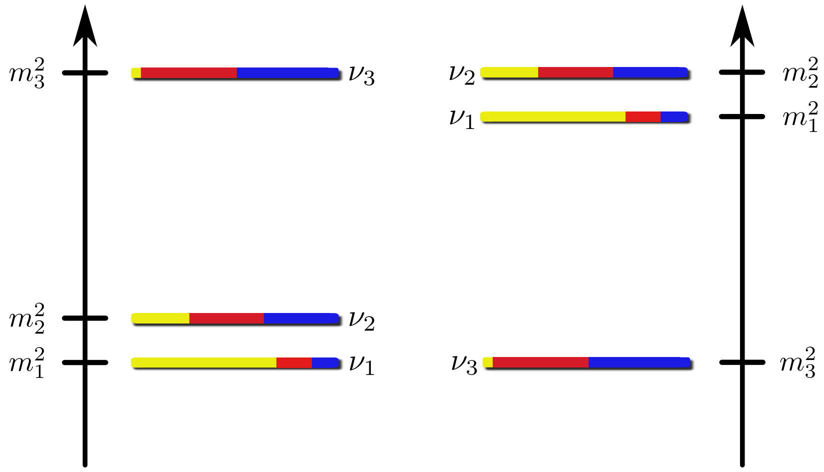


Figure 1: Sketch of the two possible neutrino mass hierarchies

- 2 detectors at 470 m
- 1 detector at 576 m
- 3 detector at 1.65 km

$$\theta_{13} = 8.8^\circ$$

6.14 Neutrino mass hierarchy and Majorana neutrinos

We know that $\Delta m^2_{21} \ll |\Delta m^2_{31}| \simeq |\Delta m^2_{32}|$. Two possible hierarchies. Δm^2_{21} is known to be positive. MSW effect in the sun provides sensitivity on the sign (note: standard oscillation formula is insensitive to sign).

- Need (at least) one direct mass measurement
- e.g. effective electron-neutrino mass from beta-decays

$$m_e = \sum_{i=1}^{3} \left| U^2_{ei} m_i \right|$$

Current best limits:

$$m_e < 2\,\text{eV}$$

KATRIN (Karlsruhe @KIT) aims for limit 0.2 eV. Alternative: MSW effect in $\nu_\mu \to \mu_\tau$ oscillations (ANGU/ORCA). From cosmological measurement (Planck):

$$\sum_{m_n} < 0.2\,\text{eV}$$

Another important question: Is there a difference between ν and $\overline{\nu}$? For charged fermions this is clear:

$$\hat{C}\left|e^-\right\rangle = \left|e^-\right\rangle$$

For neutrinos:

$$\hat{C}\left|\nu_e\right\rangle = \left|\overline{\nu_e}\right\rangle$$

if neutrinos are Dirac particles, alternative:

$$\hat{C}\left|\nu_e^\mu\right\rangle = \left|\mu_e^\mu\right\rangle$$

if neutrinos are Majorana particles, i.e.:

$$\left|\nu\right\rangle \equiv \left|\overline{\nu}\right\rangle \equiv \left|\nu^\mu\right\rangle$$

Dirac case:

$$\left.\begin{matrix}\nu_L\\ \overline{\nu_R}\end{matrix}\right\} \text{ take part in weak interactions}$$

$$\left.\begin{matrix}\overline{\nu_L}\\ \nu_R\end{matrix}\right\} \text{ Do not take part in any interaction (but gravity) if the exist}$$

Majorana case:

$$\left.\begin{matrix}\nu_L^\mu\\ \nu_R^\mu\end{matrix}\right\} \text{ Take part in weak interactions}$$

Can these two possibilities be distinguished? Consider $m_\mu = 0$. Fact: β^+/β^--decays produce two different kinds of neutrinos. Standard scenario:

$$\beta^-\text{-decay } n \to p + \overline{e_L \nu_{e,R}}$$

Detection:

$$\overline{\nu_{e,R}} + W^+ \to e_R^+$$

$\to$ Neutrino from β^--decay always produces a positron, never an electron. β^+-decay: $p \to n + e_R^+ + \nu_{e,L}$. Detection: $\nu_{e,L} + W^- \to \overline{e_L}$. Neutrino from β^+-decay always produces an electron, never a positron.

Proof for existence of ν and $\overline{\nu}$ as different particles? No: $\nu_{e,L}$ and $\overline{e,R}$ could just be one particle with two different chiralities (Majorana's proposal).

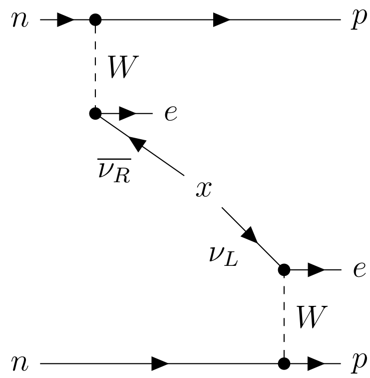
$$\nu_{e,L}^\mu \equiv \nu_{e,L} \text{ Only couples to } e^-$$

$$\nu^{\mu}_{e,R} \equiv \overline{\nu_{e,R}} \text{ Only couples to } e^+$$

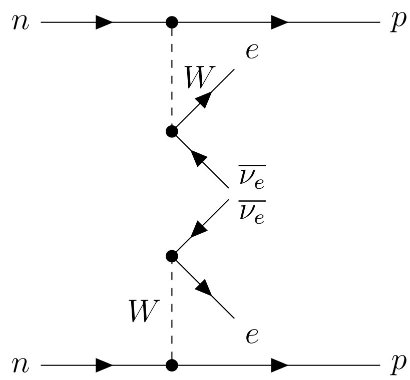
$\rightarrow$ Complies $\Delta L_e = 2$ i.e. lepton number violation. $\rightarrow$ No chance to distinguish experimentally since identical signature. Mass $m_\nu \neq 0$, chirality $\neq$ helicity.

$$\nu_L = \nu_\downarrow + \epsilon \cdot \nu \quad \epsilon \ll 1 \nu_R = \nu_\uparrow + \epsilon \nu_\downarrow$$

for an initial ν_L small probability to later react as ν_R. Most promising detection method: Neutrinoless double beta decay (Or $\beta\beta$).

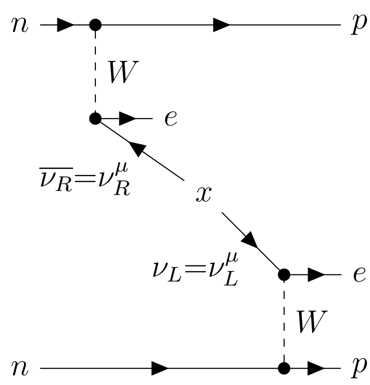


Double-β decay ("$2\nu\beta\beta$")



- Two simultaneouse β decays in the same nucleus
- Observable in some nuclei for which single β-decay is energetically forbidden

Neutrinoless double-β decay ($0\nu\beta\beta$):



No neutrino in final state, two monochromatic electrons only. Prerequisites:

- $\nu^\mu \equiv \nu \equiv \overline{\nu}$ (Majorana scenario, $\Delta L = 2$)
- $m_\mu \neq 0$ (chirality $\neq$ helicity)
- Currently best experiments use Germanium:

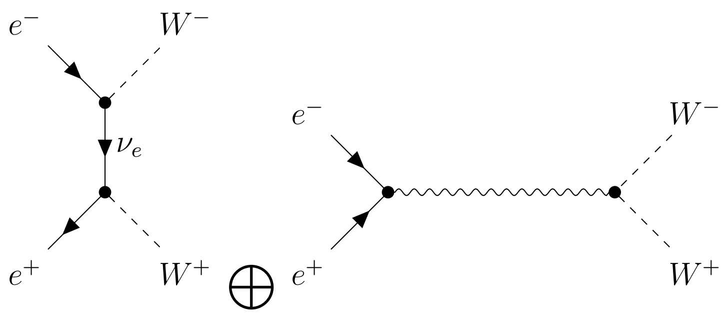
$$^{76}Ge \to ^{76} Se + 2e^-$$

- Decay material = detector (semiconductor, excellent energy resolution)
- Problem: $\tau(2\nu\beta\beta) \sim 10^{21}$yr
- Experiment GERDA: $m \sim 20\,\mathrm{kg}$
  No signal $\to \tau(0\nu\beta\beta) > 2 \times 10^{25}$yr
- Alternative: EXO (^{136}Xe)

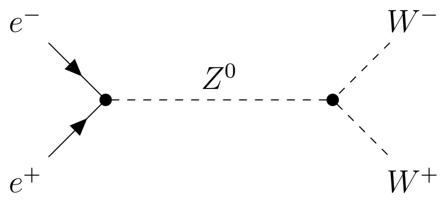
7 Neutral weak currents and electroweak unification

There is strong theoretical motivation for existence of a weak neutral current (Z^0 exchange).

Example 7.1. W production in e^+e^- collisions



With only these two contributions, the calculated cross section diverges as CM energy increases. → unphysical. Additional weak neutral current cancles parts of the other amplitudes:



→ Finite ross section.

7.1 Concept of local gauge invariance

QED, QCD and weak interaction are fundamentally based on principle that physics be invariant under local phase transformation of the wave function.

QED:

$$\psi(x) \to \psi'(x) = e^{iq\chi(x)}\psi(x)$$

a U(1) symmetry. Dirac eq. (free particle):

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi = m\psi \quad \to \quad & i\gamma^\mu \partial_\mu (e^{iq\chi(x)}\psi) = m e^{iq\chi(x)}\psi \\ & \Rightarrow i\gamma^\mu \left[\partial_\mu \psi + iq(\partial_\mu \chi)\psi\right] = m\psi \end{aligned}$$

Not invariant under local phase transformation. ⇒ Need to introduce interaction term to Diract eq. (or lagrangian):

$$i\gamma^\mu \left(\partial_\mu + iqA_\mu\right)\psi = m\psi$$

A: Field corresponding to massless gauge boson (i.e. photon). → Invariances restored if

$$A_\mu \to A'_\mu = A_\mu \partial_\mu \chi$$

→ "automatically" produced electron-photon interaction.

QCD:

$$\psi(x) \to \psi'(x) = \exp\left(ig_s \vec{\alpha}(x) \cdot \vec{T}\right)\psi(x)$$

the SU(3) symmetry with $\vec{T} = (T1, T2, \cdots, T_8$, eight generators of the SU(3) symmetry groupand $\vec{\alpha}(x) = (\alpha_1, \alpha_2, \cdots, \alpha_8$, the local phase. Local phase invariance → symmetry of QCD w.r.t. color of the particle ⇒ Need to produce 8 interaction terms with 8 new gluon fields. $\vec{G_\mu} = (G^1_\mu, G^2_\mu, \cdots, G^8_\mu$:

$$i\gamma^\mu \left(\partial_\mu + \underbrace{ig_s \vec{G_\mu} \cdot \vec{T}}_{\text{interaction of particles with fields}}\right)\psi = m\psi$$

→ Invariance restored if

$$G^k_\mu \to G^{k'}_\mu = G^k_\mu - \partial_\mu \alpha^\mu - g_s f^{ijk} \alpha_i G_{\mu,j}$$

Weak interaction

$$\psi_L(x) \to \psi'_L(x) = \exp\left(i g_w \vec{\alpha}(x) \cdot \vec{T}\right) \psi_L(x)$$

with

$$\begin{aligned} T_i &= \frac{1}{2}\sigma_i = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ T_2 &= \frac{1}{2}\sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ T_3 &= \frac{1}{2}\sigma_e = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

These are the generators of the SU(2) group symmetry and $\vec{\alpha}(x)$ the local phase. Local phase invariance → Symmetry w.r.t. *weak isospin* of the particle.

⇒ Need to add 3 interation terms with 3 boson fields. $W^i_\mu, i = 1, 2, 3$:

$$i\gamma^\mu \left(\partial_\mu + \underbrace{i g_w \vec{W}_\mu \cdot \vec{T}}_{\text{interaction of particles w. field}}\right) \psi_L = m\psi_L$$

Write wave function $\psi_L(x)$ as *isospin-doublet* of left-handed fermion spinors:

$$\psi_L = \begin{pmatrix} \nu_e(x) \\ e^-(x) \end{pmatrix}_L$$

In terms of weak isospin I_w:

$$\begin{aligned} \left|\nu_e\right\rangle_L &= \left| I_w = \frac{1}{2}, I_w^{(3)} = +\frac{1}{2} \right\rangle \\ \left|e^-\right\rangle_L &= \left| I_w = \frac{1}{2}, I_w^{(3)} = -\frac{1}{2} \right\rangle \end{aligned}$$

- A charged weak interaction is a rotation in weak isospin space, leaving I_w unchanged.
- Local gauge invariance ensures that weak interaction is invariant w.r.t. $I_w^{(3)}$
- Analogon: Spin states of a spin-$\frac{1}{2}$ particle:

$$\begin{aligned} \left|\uparrow\right\rangle &= \left| s = \frac{1}{2}, m_s = +\frac{1}{2} \right\rangle \\ \left|\downarrow\right\rangle &= \left| s = \frac{1}{2}, m_s = -\frac{1}{2} \right\rangle \end{aligned}$$

Weak isospin doublets for all fermions:

$$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L \quad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L \quad \underbrace{\begin{pmatrix}u\\ d'\end{pmatrix} \quad \begin{pmatrix}c\\ s'\end{pmatrix}_L \quad \begin{pmatrix}t\\ b'\end{pmatrix}_L}_{\text{weak eigenstate}}$$

+singlets $e_R^-, \mu_R^-, \tau_R^-, u_R, d_R', c_R, s_R', t_R, b_R'$. We have introduced 3 boson fields with interactions → 3 fermion currents.

$$j_k^\mu = g_w \overline{\psi_L}\gamma^\mu \, \mathrm{T}_k \, \psi_L$$

coupling of doublet ψ_L with W_k^μ. Physical W^+ and W^- bosons are linear combinations of W_μ^1 and W_μ^2:

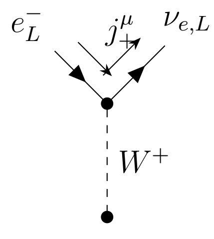
$$W_\mu^\pm(x) = \frac{1}{\sqrt{2}}\left[W_\mu^1(x) \pm i W_\mu^2(x)\right]$$

Charged weak currents:

$$\begin{aligned} j_+^\mu &= \frac{g}{\sqrt{2}}\overline{\psi_L}\gamma^\mu(T_1 + iT_2)\psi_L \\ &= \frac{g}{\sqrt{2}} = \overline{\psi_L}\gamma^\mu\frac{1}{2}(\sigma_1 + i\sigma_2)\psi_L \\ &= \frac{g}{\sqrt{2}}\overline{\psi_L}\gamma^\mu\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\psi_L \end{aligned}$$

For the eletron doublet:

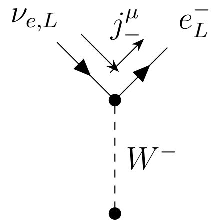
$$\begin{aligned} j_+^\mu = \frac{g}{\sqrt{2}}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \gamma^\mu \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \\ = \frac{g}{\sqrt{2}}(\overline{\nu_e})_L\gamma^\mu e_L^- \\ = \frac{g}{\sqrt{2}}\overline{\nu_e}\gamma^\mu\frac{1}{2}(1-\gamma^5)e^- \end{aligned}$$



with $\frac{g}{\sqrt{2}}$ in the vertex. Charge-raising current:

$$j_-^\mu = \frac{g}{\sqrt{2}}\overline{\psi_L}\gamma^\mu(T_1 - iT_2)\psi_L = \frac{g}{\sqrt{2}}\overline{\psi_L}\gamma^\mu\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\psi_L$$

$$= \frac{g}{\sqrt{2}}(\overline{e_L^-})\gamma^\mu \nu_{e,L} = \frac{g}{\sqrt{2}}\overline{e^-}\gamma^\mu \frac{1}{2}\left(1-\gamma^5\right)\nu_2$$



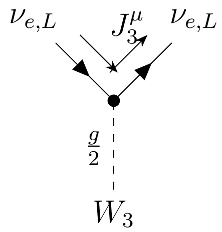
Charge lowering current. $\rightarrow$ SU(2)$_L$ symmetry automatically delivers femion couplings to W^+/W^- bosons.

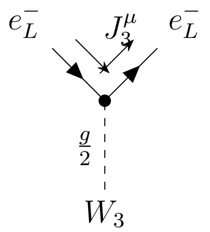
7.2 Electroweak unification

There is one field missing: W_μ^3 left but predicted by $SU(2)_L$. Corresponding Fermion current:

$$\begin{aligned} J_3^\mu = g\overline{\psi_L}\gamma^\mu T^3 \psi_L &= \frac{g}{2}\begin{pmatrix}\overline{\nu_e} \\ e^-\end{pmatrix}\gamma^\mu \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L \\ &= \frac{g}{2}\overline{\nu_{e,L}}\gamma^\mu \nu_{e,L} - \frac{g}{2}\overline{e_L^-}\gamma^\mu e_L^- \end{aligned}$$

Left handed coupling between leptons of the same electric charge $\rightarrow$ neutral current.





Is the W^3 the Z^0? Difficult:

- $m(Z^0)$ not similar to $m(W^\pm)$
- W^3 couples strictly to left-handed particles

$$J_3^\mu = \psi_L \gamma^\mu T_3 \psi_L = \overline{\psi}\gamma^\mu \frac{1}{2}(1-\gamma^5)T_3\psi$$

$$T_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

$\rightarrow I_W^{(3)}$ defines the sign for $I_W^{(3)} = 0$: No interaction. But experimentally: Z^0 couples partially to right-handed particles.

$$J_{NC}^\mu \sim \overline{\psi}\gamma^\mu \frac{1}{2}(c_V - c_A\gamma^5)\psi$$

with $c_A \neq c_A \neq 1$

How can the idea of the $SU(2)_L$ symmetry be kept?

The Electroweak GSW model (Glashow, Salam, Weinberg). Photon and Z^0 are both neutral. Both couple to left- and right-handed particles. → Are these fields A_μ and Z_μ superpositions of more fundamental fields B_μ and W_3^μ?

$$\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B^\mu \\ W_3^\mu \end{pmatrix}$$

with θ_W: The Weinberg angle. Combine W_3 from $SU(2)_L$ with the B_μ from $U(1)$ into two bosons → Electroweak unification.

- B_μ: New gauge field proposed to couple to the hypercharge Y of the fermions

It is now the interaction with B_μ that is naturally obtained by requirement of local $U(1)$ gauge invariance (rather than A_μ).

- Old:

$$\psi(x) = \psi(x)\exp\{iq\xi(x)\} \qquad U(1)_{\text{em}}$$

  → Coupling to A_μ with strength q. Remember: Interaction $-iqJ_{\text{em}^\mu} \cdot A_\mu$

- New:

$$\psi'(x) = \psi(x)\exp\left\{ig'\frac{Y}{2}\phi(x)\right\} \qquad U(1)_Y$$

  → Coupling to B_μ with strength $g'\frac{Y}{2}$. Interaction:

$$g'\frac{Y}{2}\gamma^\mu B_\mu = g'\frac{Y}{2}J_Y^\mu B_\mu$$

Consequence: B_μ and W_μ^3 are fundamental boson fields of nature. A^μ and Z^μ are observable fields (via bosons γ and Z^0).

Why? Electroweak symmetric breaking (Higgs mechanism). Related fundamental currents:

$$W_\mu^3 : J_3^\mu = \frac{g}{2}\overline{\nu_L}\gamma^\mu\nu_L + \frac{g}{2}\overline{e_L}\gamma^\mu e_L$$

$$B_\mu : J_Y^\mu = \frac{g'}{2}Y_{\nu_L}\overline{\nu_L}\gamma^\mu\nu_L + \frac{g'}{2}Y_{\nu_R}\overline{\nu_R}\gamma^\mu\nu_R + \frac{g'}{2}Y_{e_L}\overline{e_L}\gamma^\mu e_L + \frac{g}{2}Y_{e_R}\overline{e_R}\gamma^\mu e_R$$

Right-handed and left-handed couplings have different strengths. With hypercharge:

$$Y = 2(Q - I_W^{(3)})$$

Q: Electric charge, $I_W^{(3)}$ of isospin. Reason:

1. Hypercharge in weak isospin doublet should be the same (otherwise $U(1)_Y$ local gauge transformation would induce a phase shift in the two components of the doublet, breaking the $SU(2)$ symmetry)

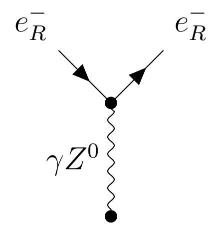
2.
$$
\begin{aligned}
Y &= \alpha Q + \beta I_W^{(3)} \\
Q(e_L^-) = -1 \; ; \; I_W^{(3)}(e_L^-) &= -\frac{1}{2} \\
Q(\nu_{e_L}) = 0 \; ; \; I_W^{(3)}(\nu_{e_L}) &= +\frac{1}{2} \\
Y_{e,L} \stackrel{!}{=} Y_{\nu_{e,L}} \Rightarrow -\alpha - \frac{1}{2}\beta \stackrel{!}{=} \frac{1}{2}\beta \Rightarrow \beta \stackrel{!}{=} -\alpha \Rightarrow Y &= -I_W^{(3)}
\end{aligned}
$$

For the Electron/electron-neutrino:

	I_W	I_W^3	Q	Y
ν_L	$\frac{1}{2}$	$+\frac{1}{2}$	0	-1
e_L^-	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
e_R^-	0	0	-1	-2
ν_R	0	0	0	0

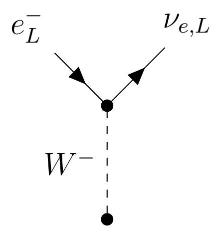
Similar for other fermions. Important: By requesting local gauge invariance both under $U(1)_Y$ and $SU(2)_L$ all electroweak interaction (neutral and charged) conserve both Y and I_W at each vertex simultaneously.

Example 7.2.



Before: $Y = -2, I_W = 0$, after: $Y = -2, I_W = 0$

Example 7.3.



Before: $Y = -1, I_W = \frac{1}{2}, I_W^3 = -\frac{1}{2}$, after: $Y = -1, I_W = \frac{1}{2}, I_W^3 = +\frac{1}{2} \rightarrow$ rotation in I_W space.

Observed physical currents:

- QED current:

$$J^\mu_{\text{em}} = J^\mu_Y \cos\theta_W + J^\mu_3 \sin\theta_W$$

  (photon exchange)

- Neutral weak current:

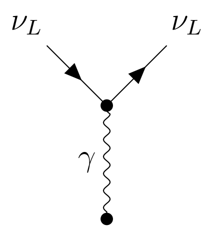
$$J^\mu_{NC} = -J^\mu_Y \sin\theta_W + J^\mu_3 \cos\theta_W$$

  (Z^0 exchange)

- Charged weak currents: J^μ_+, J^μ_- (as before)

Look at a few specific processes:

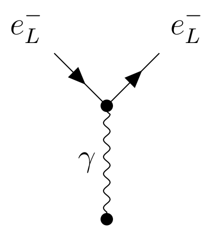
1.



e.m. current:

$$\begin{aligned} J^\mu_{\text{em}} &= Q(\nu) e \overline{\nu_L} \gamma^\mu \nu_L = 0 \\ &\stackrel{!}{=} \frac{g'}{2} Y_{\nu_L} \gamma^\mu \nu_L \cos\theta_W + \frac{g}{2} \overline{\nu_L} \sin\theta_W \\ &\to g \cdot \sin\theta_W = g' \cdot \cos\theta_W \end{aligned}$$

2.



$$\begin{aligned} J^\mu_{\text{em}} &= -e \overline{e_L} \gamma^\mu e_L \\ &\stackrel{!}{=} \frac{g'}{2} Y_{e,L} \overline{e_L} \gamma^\mu e_L \cos\theta_W - \frac{g}{2} \overline{e_L} \gamma^\mu e_L \sin\theta_W \\ &\to e = g \cdot \sin\theta_W = g' \cos\theta_W \end{aligned}$$

Relation between e.m. and weak coupling constants (g, g') → another way of expressing the electroweak unification. Prediction (Higgs mechanism):

$$\cos\theta_W = \frac{m_W}{m_Z}$$

$\simeq 0.88$ from experiment: $(\sin\theta_W)^2 = 0.23$.

7.3 Coupling of Z to fermions

Z^0 couples to both, left- and right-handed fermions since

$$J^\mu_{NC} = -\frac{g'}{2}\underbrace{[Y_{f_L}\overline{u_L}\gamma^\mu u_L + Y_{f_R}\overline{u_R}\gamma^\mu u_R]}_{\text{coupling to } B_\mu}\sin\theta_W + \underbrace{I_W^{(3)} g\overline{u_L}\gamma^\mu u_L \cos\theta_W}_{\text{coupling to } W^3_\mu}$$

with $g'\cos\theta_W = g\sin\theta_W$ and $Y = 2(Q - I^3_W)$

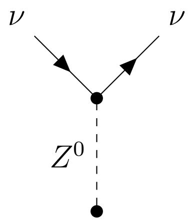
$$\to J^\mu_{NC} = \frac{g}{\cos\theta_W}[c_L\overline{u_L}\gamma^\mu u_L + c_R\overline{u_R}\gamma^\mu u_R]$$

with

$$c_L = I^3_W - Q\sin\theta_W$$
$$c_R = -Q\sin\theta_W$$

Coupling strength depends on type of fermion (I^3_W, W) and its chirality (L/R).

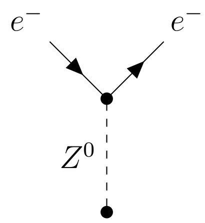
Example 7.4. Neutrino ($\nu_L : I_W = \frac{1}{2}; I_W^{(3)} = +\frac{1}{2}; Q = 0$ $\nu_R : I_W = 0; I_W^{(3)} = 0; Q = 0$)



$$J^\mu_{NC} = \frac{g}{\cos\theta_W}(\frac{1}{2} - 0)\overline{u_L}\gamma^\mu u_L + \frac{g}{\cos\theta_W}(0)\overline{u_R}\gamma^\mu u_R$$
$$= \frac{1}{2}\frac{g}{\cos\theta_W}\overline{u_L}\gamma^\mu u_L$$

→ Only left-handed coupling with strength $\frac{1}{2}$.

Example 7.5. Electron



$$J^\mu_{NC} = \frac{g}{\cos\theta_W}(-\frac{1}{2} + (\sin\theta)^2)\overline{e_L}\gamma^\mu e_L + \frac{g}{\cos\theta_W}\underbrace{(\sin\theta_W)^2}_{0.23}\overline{e_R}\gamma^\mu e_R$$
$$= \frac{g}{\cos\theta_W}\left[-0.27\overline{e_L}\gamma^\mu e_L + 0.23\overline{e_R}\gamma^\mu e_R\right]$$

$\rightarrow$ Almost equal strength for left-handed and right-handed coupling (strength ~ 0.25) similar for other fermions

Remark.

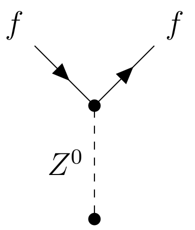
$$(\nu_e, \nu_\mu, \nu_\tau), (e^-, \mu^-, \tau^-), (u, c, t), (d, s, b)$$

coupling to Z with the same relative strength.

Coupling to Z boson can also be expressed in terms of vector and axial vector coupling (more common). Remember:

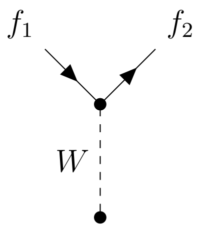
$$\overline{u_L}\gamma^\mu u_L = u\gamma^\mu \frac{1}{2}(1-\gamma^5)u$$
$$\overline{u_R}\gamma^\mu u_R = u\gamma^\mu \frac{1}{2}(1+\gamma^5)u$$
$$J^\mu_{NC} = \frac{g}{\cos\theta_W}\left[c_L\overline{u_L}\gamma^\mu u_L + c_R\overline{u_R}\gamma^\mu u_R\right]$$
$$= \frac{g}{\cos\theta_W}\overline{u}\gamma^\mu\left[c_L\frac{1}{2}(1-\gamma^5) + c_R\frac{1}{2}(1+\gamma^5)\right]u$$
$$= \frac{g}{\cos\theta_W}\overline{u}\gamma^\mu\left[(c_L + c + R) - (c_L = c_R)\gamma^5\right]u$$
$$J^\mu_{NC} = \frac{g}{\cos\theta_W}\frac{1}{2}\overline{u}\gamma^\mu(c_V - c_A\gamma^5)u$$
$$c_V = c_L + c_R = I_W^{(3)} - 2Q(\sin\theta_W)^2$$
$$c_A = c_L - c_R = I_W^{(3)}$$

Feynman rule for the Z-boson coupling:



$$-i\frac{g}{2\cos\theta_W}\gamma^\mu(c_V - c_A\gamma^5)$$

Remeber: Charged weak current:



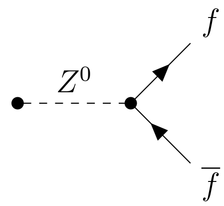
$$J^\mu_{CC} = \frac{g}{\sqrt{2}}\overline{u}\gamma^\mu\frac{1}{2}(1-\gamma^5)u$$

(strictly left-handed coupling, $c_V = c_A = 1$)

Remark. If θ_W known (by measurement) all couplings of the Z-boson are fixed (by $I_W^{(3)}$ and Q of the particle). $\rightarrow$ Large preditive power

Remark. Beause the weak neutral current contains both, vector and axial coupling, it does not conserve parity.

Example 7.6. Z^0 decay widht.



Prediction: $\Gamma(Z^0 \rightarrow f\overline{f}) = \frac{1}{48\pi} \frac{g^2}{(\cos\theta_W)^2}(c_V^2 + c_A^2) \cdot m_Z$

$$\left.\begin{aligned} c_V &= I_W^3 - 2Q(\sin\theta_W)^2 \\ c_A &= I_W^3 \end{aligned}\right\} \text{are defined by choice of } f\overline{f}$$

Example 7.7. $Z^0 \rightarrow \nu_e\overline{\nu_e}$ $c_V^2 = \frac{1}{4} = c_A^2$

$$\rightarrow \Gamma(Z^0 \rightarrow \nu_e\overline{\nu_e}) = 167\,\text{MeV}$$

Total decay width:

$$\Gamma_{\text{tot}} = \sum_f \Gamma(Z^0 \rightarrow f\overline{f})$$

with $f \in \{\nu_e, \nu_\mu, \nu_{tau}, e^-, \mu^-, \tau^-, u, d, c, s, b\}$ (t is to heavt).

$$\rightarrow \Gamma_{\text{tot}} = 2.5\,\text{GeV}$$

Predicted branching ratios:

$$BR(Z^0 \rightarrow \begin{Bmatrix} e^+e^- \\ \mu^+\mu^- \\ \tau^+\tau^- \end{Bmatrix}) = \frac{\Gamma(Z^0 \rightarrow \begin{Bmatrix} e^+e^- \\ \mu^+\mu^- \\ \tau^+\tau^- \end{Bmatrix})}{\Gamma_{\text{tot}}} \simeq 3.5\,\%$$

Experimentally:

$$\begin{Bmatrix} 3.364\,\% \\ 3.366\,\% \\ 3.370\,\% \end{Bmatrix}$$

$$BR(Z^0 \to 2\text{jets}) = \frac{\sum_{u,d,s,c,b} \Gamma(Z^0 \to q\overline{q})}{\Gamma_{\text{tot}}} \simeq 70\,\% \quad (\text{Exp:}69.9\,\%)$$

$$BR(Z^0 \to \nu\overline{\nu}) = \frac{\sum_{\nu_e,\nu_\mu,\nu_\tau \Gamma(Z^0 \to \nu \overline{nu})}}{\Gamma_{\text{tot}}} = 20\,\%$$

$\to$ Only 3 (light) neutrino flavours.

8 Gauge symmetries and Higgs mechanism

Lagrangians in QFT: Connection between symmetries and conservation laws is best described in Lagrangian fied theory.

1. Classical:

$$L = T - V = L(q_i, \dot{q}_i)$$

   Euler-Lagrangian equation describes the equation of motion of a particle:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

   Example 8.1. Particle moving in one dimension where Lagrangian is function of the space coordiante x and its time derivative $\dot{x}$.

$$L = T - V = \frac{1}{2} m\dot{x}^2 - V(x)$$
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \frac{\partial L}{\partial x} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

   E-L-equation:

$$m\ddot{x} = \frac{-\,\mathrm{d}V(x)}{\mathrm{d}x} = F$$

   (Newton's law)

2. Continous Lagrange-density $\mathcal{L}$ $L = \int \mathcal{L}\, \mathrm{d}^3x$

$$L(q_i, \dot{q}_i) \to \mathcal{L}(\Phi_i, \partial_\mu \Phi_i)$$

   Euler-Lagrangian equation:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi_i)}\right) - \frac{\partial \mathcal{L}}{\partial \Phi_i} = 0$$

   - Lagrange density describes dynamics of the quantum field $\Phi(x)$

- Particles: Excitations of this quantum field
- E-L: Describes equation of motion of $\Phi(x)$

Example 8.2. Free scalar boson field (Spin= 0). If we have a Lagrangian of the following form:

$$\mathcal{L}_{\mathcal{KG}} = \underbrace{\frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi)}_{\text{kinematic term}} - \underbrace{\frac{1}{2}m^2\Phi^2}_{\text{mass-term}}$$
$$\rightarrow \frac{\partial L}{\partial \Phi} = m^2\Phi, \dots \rightarrow \text{ in E-L-equ.}$$
$$\partial_\mu \partial^\mu \Phi(x) + m^2\Phi(x) = 0$$

the Klein-Gordon equation

Example 8.3. Free spin $\frac{1}{2}$ fermion field $\psi(x)$

$$\mathcal{L}_{\mathcal{D}} = i\overline{\psi}\gamma_\mu \partial^\mu \psi - m\psi\overline{\psi}$$
$$\rightarrow \frac{\partial \mathcal{L}_{\mathcal{D}}}{\partial \psi} \rightarrow \text{ E-L-eq.}$$
$$\Rightarrow i\gamma_\mu \partial^\mu \psi(x) - m\psi(x) = 0$$

The Dirac-equation

Example 8.4. Vector boson field

$$\mathcal{L}_{\mathcal{EM}} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}}_{\text{kinematis term}} \underbrace{-j^\mu A_\mu}_{\text{sources term}} \underbrace{+\frac{1}{2}m_\mu^2 A^\mu A_\mu}_{\text{mass term}}$$
$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \dots \quad \text{E-L-eq. } \partial_\mu$$
$$\Rightarrow F^{\mu\nu} = j^\nu$$

Maxwell-equations with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ (Lorentz-gauge)

8.1 QED and local gauge invariance

Postulate QED invariant w.r.t. $U(1)$ local phase transformation (=local gauge symmetry).

$$\psi(x) \rightarrow \psi'(x) = e^{iq\chi(x)}\psi(x)$$

where $\chi(x)$ is a scalar funtion of x.

Remark. $\overline{\psi}(x) \rightarrow \overline{\psi'}(x) = e^{-q\chi(x)}\overline{\psi(x)}$

Is $\mathcal{L}_D$ unchanged by this transformation? Not quite:

$$\mathcal{L}_D \rightarrow \dot{\mathcal{L}}_D = i\overline{\psi}e^{-q\chi(x)}\gamma^\mu \partial^\mu \left(\psi e^{+iq\chi(x)}\right) - m\psi\overline{\psi}$$

$$= -i\overline{\psi}e^{-iq\chi(x)}\gamma_\mu \left(e^{-iq\chi(x)}\partial^\mu\psi + iqe^{iq\chi(x)\psi\partial^\mu\chi(x)}\right) - m\psi\overline{\psi}$$

$$= \underbrace{i\overline{\psi}\gamma_\mu\partial^\mu\psi - m\psi\overline{\psi}}_{\mathcal{L}_D} - \underbrace{q\overline{\psi}\gamma_\mu\psi\partial^\mu\chi(x)}_{\neq 0 \text{if} \chi=\chi(x)}$$

Local gauge invariance of $\mathcal{L}_D$ is saved if

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu(x)$$

Covariant derivative, where at the same time

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi(x)$$

$\to$ used to cancel the unwanted term in $\mathcal{L}'_D$. A_μ: Gauge field. $\to U(1)$ local gauge invariance of the Diract Lagrangian can only be achieved by introducing

$$A_\mu$$

New Lagrangian: (replace $\partial^\mu \to D^\mu$):

$$\mathcal{L}_D = i\overline{\psi}\gamma_\mu D^\mu\psi - m\psi\overline{\psi}$$

$$= \underbrace{i\overline{\psi}\gamma^\mu\partial^\mu\psi}_{\text{kinematics term}} - \underbrace{m\overline{\psi}\psi}_{\text{mass term}} - \underbrace{q\overline{\psi}\gamma_\mu\psi A^\mu}_{\text{interaction term}}$$

Now

$$\mathcal{L}'_D = \mathcal{L}$$

Local gauge principle provides an elegant description of the interaction

$$q\overline{\psi}\gamma_\mu\psi A^\mu = j_\mu^{\text{em}} A^\mu$$

$\to$ Automatically generated the interaction of a fermion with el. charge q with photon filed by requirement of local $U(1)$ gauge invariance. For full QED Lagrangian (describing filed of fermion, of the photon and their interaction): Add kinetic energy of photon field. Full QED Lagrangian for the eletron ($q = -e$):

$$\mathcal{L}_D = \underbrace{i\overline{\psi}\gamma^\mu\partial_\mu\psi - m_e\overline{\psi}\psi}_{\text{field for electron}} + \underbrace{e\overline{\psi}\gamma^\mu\psi A_\mu}_{\text{interaction}} + \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{fied for massless photon}}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the e.m. field strength tensor.

1. Solve Euler-Lagrange equation for $psi(x) \Rightarrow$ Wave equation of electron in e.m. potential A_μ (Dirac-equation)

2. Solve Euler-Lagrange equation for $A_\mu(x) \to$ Maxwell's equations

Similarly for the $SU(2)_L \times U(1)_Y$ electroweak Lagrangian but for complicated (W_μ and B_μ fields). For $SU(2)_L : \partial_\mu \to D_\mu = \partial_\mu + ig\vec{T}\vec{W}_\mu(x)$.

8.2 Mass of the photon

Could we add a mass term to QED Lagrangian that makes the photon (field A_μ) massive.

$$\mathcal{L}_{\text{QED}} \to \mathcal{L}_{\text{QED}} + \frac{1}{2}m_\mu^2 A_\mu A^\mu$$

If the photon would have mass this would be the term to be added. Because of the local gauge invariance the photon fields transforms as:

$$\begin{aligned} A_\mu \to A'_\mu &= A_\mu - \partial_\mu \chi(x) \\ \Rightarrow \frac{1}{2}m_\mu^2 A_\mu A^\mu \to \frac{1}{2}m_\mu^2 (A_\mu - \partial\chi)(A^\mu - \partial^\mu\chi) &+ \frac{1}{2}m_\mu^2 A_\mu A^\mu \end{aligned}$$

$\to$ Mass-term is forbidden by requirement of local gauge invariance. $U(1)$ local gauge symmetry can only be satisfied if the gauge photon of the interaction is massless (Not restricted to $U(1)$).

8.3 Spontaneous symmetry breaking and the Higgs-mechanism

Need to introduce mass terms for $W^\pm$ and Z^0 without breaking the local gauge symmetry. This can be done by introducing a new scalar boson field $\Phi(x)$ that couples to the vector fields W_μ^3 and B_μ. For simplicity shown here for the vector field A_μ and breaking of the $U(1)$ symmetry. New field $\phi(x)$ is a complex scalar field

$$\Phi(x) = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$$

$\Phi(x)$ provides a special potential

$$V(\Phi) = \mu^2 \Phi^*\Phi + \lambda(\Phi^*\Phi)^2$$

Gauge-invariant Lagrangian for field Φ:

$$\begin{aligned} \mathcal{L} &= (\partial_\mu \Phi)(\partial^\mu \Phi) - V(p) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ \mathcal{L} &= \underbrace{(\partial_\mu - iqA_\mu)\Phi^* (\partial^\mu + iqA^\mu)\Phi(x)}_{1} - \underbrace{\mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2}_{2} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{3} \end{aligned}$$

1. The kinetic energy of Φ and interaction with A_μ with coupling strength q.
2. Potential of Φ
3. Kinetic energy of massless vector field A_μ

This Lagrangian is invariant w.r.t. $U(1)$ local phase transformation ($\Phi^*\Phi = \Phi'^*\Phi'$). For $\mu^2 > 0$: $-\mu^2\Phi^*\Phi$ can be interpreted as mass term for Φ with mass $M_\Phi = \sqrt{2\mu^2}$. $(\Phi^*\Phi)^2$: Self-interaction of scalar field. Consider the case $\mu^2 < 0$:

$$V(\Phi) = \mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2$$

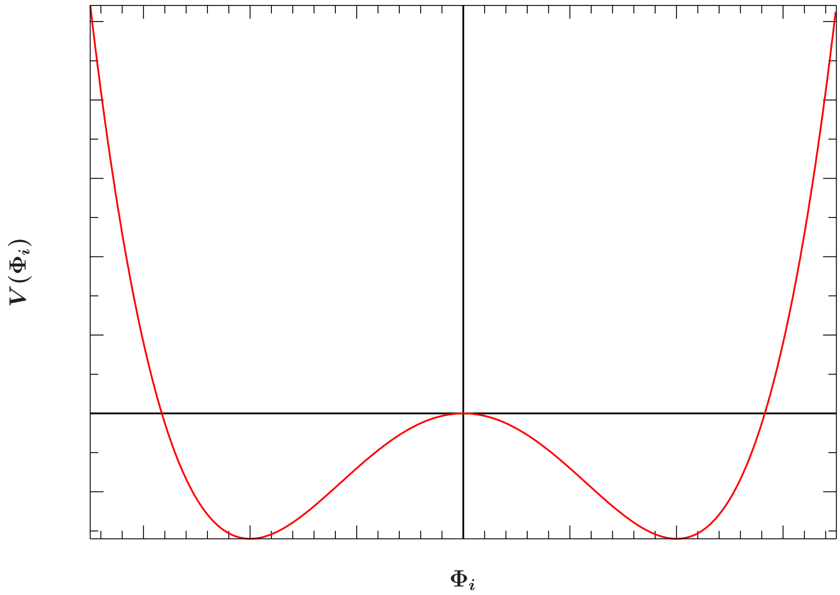


Figure 2: Sketch of the "mexican hat" potential, $\mu^2 < 0; \lambda > 0$, rotational symmetric

Vacuum state is the lowest energy state of field Φ (QFT). $\Rightarrow \lambda$ has to be positive. The circle of minima is given by the radius

$$v = \sqrt{\frac{-\mu^2}{\lambda}}$$

Lowest state is not at $\Phi = 0$. $\Rightarrow$ Potential symmetric w.r.t. rotations in the Φ_i plane $\rightarrow$ unchanged by local phase transformation of Φ. The ground state is not at $\Phi =$ but there are infinitely many ground state on the circle with radius $v = \sqrt{\frac{-\mu^2}{\lambda}}$. Nature has to choose on ground state out of infinitely many. $\Rightarrow$ Choice breaks rotational (i.e. $U(1)$) symmetry of the Lagrangian (spontaneously). Analogy: Ferromagnet Lagrangian has no preferred direction. Below Curie temperature the spins align in a particular direction and thereby breaking the symmetry. We pick

$$\Phi_1 = +v \ , \ \Phi_2 = 0$$

without loss of generality. Excitations of the field Φ (which describe the particles in QFT) can be obtained from perturbative expansion around this point:

$$\Phi(x) \simeq \frac{1}{\sqrt{2}} \left(v + \eta(x) + i\phi(x)\right)$$

- v: Vacuum expectation value (EV)
- η, ϕ: New fields along Φ_1 and Φ_2 directions

Insert this into the Lagrangian:

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta)}_{1} - \underbrace{v^2 \lambda \eta^2}_{2} + \underbrace{\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)}_{3} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{4} + \underbrace{\frac{1}{2} g^2 v^2 A_\mu A^\mu}_{5} - \underbrace{V_{\text{int}}}_{6} + \underbrace{g v A_\mu (\partial^\mu \phi)}_{7}$$

$$V_{\text{int}} = \lambda v\eta^3 + \frac{1}{4}\lambda\phi^4 + \lambda v\eta\phi^2 + \frac{1}{2}\eta^2\phi^2$$

- 1 + 2: Massive field $\eta(x)$, $m_\eta = \sqrt{2v^2\lambda}$
- 3: Massless field ϕ
- 4 + 5: Massive vector field A_μ
- 6 + 7: Interaction terms of $\eta(x), \phi(x), A_\mu(x)$ with each other

Spontaneous symmetry breaking did 3 things:

- Gauge boson field A_μ acquires a mass while Lagrangian is still invariant w.r.t. local phase transitions
- Created a new massive field $\eta(x)$ and a new massless field $\phi(x)$
- The new fields $\eta(x)$ and $\phi(x)$ interact with $A_\mu(x)$ and each other

Important: Exactly the same Lagrangian as before with the complex scalar field $\Phi(x)$ expanded around the vacuum state. $\Rightarrow$ Underlying gauge symmetry is hidden but not removed.

One poblem: Term (7) $gvA_\mu\partial^\mu$: Direct coupling of $\phi(x)$ with A_μ. Transforming a spin-0 scalar field into a spin-1 vector field, which is not desired. This massless field $\phi(x)$ (Goldstone boson) is unwanted. $\phi(x)$ can be absorbed into $\eta(x)$ by a special choice of the local gauge condition. Remember:

$$\phi \rightarrow \phi' = \phi e^{ig\chi(x)}$$
$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu\chi(x)$$

leaves Lagrangian unchanged (local $U(1)$-symmetry). $\rightarrow$ allows to pick a specific function $\chi(x)$ without changing the pysics. Part of the Lagrangian involves $\phi(x)$ and A_μ:

$$\frac{1}{2}(\partial_\mu\phi)(\partial^\mu) + gvA_\mu(\partial^\mu\phi) + \frac{1}{2}g^2v^2A_\mu A^\mu$$
$$= \left[A_\mu + \frac{1}{gv}\partial_\mu\phi\right]^2$$

Choice of gauge $\chi(x) = -\frac{1}{gv}\phi(x)$ (unitary gauge):

$$\rightarrow \frac{1}{2}g^2v^2\left[A_\mu + \frac{1}{gv}\partial_\mu\phi\right]^2 = \frac{1}{2}g^2v^2A'_\mu A'^\mu$$

$\rightarrow$ i.e. $\phi(x)$ has been absorbed and does not appear in $\mathcal{L}$ anymore. Breaking of local gauge symmetry with absorption of the Goldstone boson is called *Higgs mechanism*. Rename $\eta(x)$ into $h(x)$: Higgs field. Can show that unitarity gauge means that

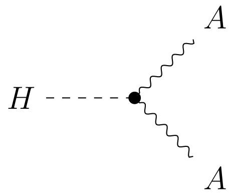
$$\Phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\phi(x)\right) \rightarrow \Phi(x) = \frac{1}{\sqrt{2}}(v + h(x))$$

The new Lagrangian in unitary gauge field $\Phi(x)$ expanded around the minimum v.

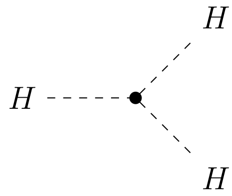
$$\begin{aligned}
\mathcal{L} &= (D_\mu\Phi)^*(D_\mu\Phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2 \\
&= \frac{1}{2}(\partial_\mu - igA_\mu)(v+h(x))(\partial_i^\mu gA^\mu)(v+h(x)) \\
&\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2(v+h(x))^2 - \frac{1}{4}\lambda(v+h(x))^2 \\
&= \frac{1}{2}(\partial_\mu h(x))(\partial^\mu h(x)) + \frac{1}{2}g^2A_\mu A^\mu(v+h(x))^2 \\
&\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda^2v^2h(x)^2 - \lambda v h(x)^3 - \frac{1}{4}\lambda h(x)^4 + \underbrace{\frac{1}{4}\lambda v^4}_{\text{const}} + \underbrace{\dots}_{\text{ignore}}
\end{aligned}$$

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu h(x))(\partial^\mu h(x)) - v^2\lambda h(x)^2}_{\text{massive Higgs field } m_H = \sqrt{2\lambda v^2}}$$

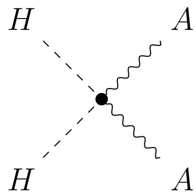
$$\underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g^2v^2A_\mu A^\mu}_{\text{massive vector field } A^\mu;\, m_A = gv} + \underbrace{vg^2A_\mu A^\mu h(x) + \frac{1}{2}g^2A_\mu A^\mu h(x)^2}_{\text{interaction of } A_\mu \text{ with Higgs field}} - \underbrace{\lambda v h(x)^3 - \frac{1}{4}\lambda h(x)^4}_{\text{Higgs self interaction}}$$



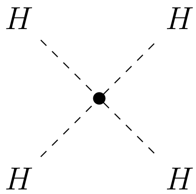
$vg^2A_\mu A^\mu$, strength vg^2



$-\lambda v h^3$, strength λv



$\frac{1}{2}g^2A_\mu A^\mu h^2$, strength $\frac{1}{2}g^2$



$-\frac{1}{4}\lambda h^4$, strength $\frac{\lambda}{4}$

Remark. The Higgs mechanism shown explained here for breaking of the (simple) $U(1)$ symmetry $\rightarrow$ Mass to the vector field A_μ

Remark. In the standard model: Used to explain the masses of the $W^\pm$ and Z^0 by breaking of $SU(2)_L \times U(1)_Y$ symmetry. $\rightarrow$ Same scheme but hav weak isospin Higgs doublet.

$$\Rightarrow \Phi(x) = \underbrace{\begin{pmatrix}\Phi^\dagger(x)\\ \Phi^0(x)\end{pmatrix}}_{\text{two complex scalar fields}} \overset{\text{breaking}}{\rightarrow} \frac{1}{\sqrt{2}}\underbrace{\begin{pmatrix}0\\ v+h(x)\end{pmatrix}}_{\text{One Higgs field}}$$

Remark. Fermion masses? Also introduced by same symmetry breaking.

8.4 Masses of SM particles

Results of symmetry breaking of $SU(2)_L \times U(1)_Y$. Lagrangian:

- Higgs mass: $m_H = \sqrt{2\lambda v^2}$
  - λ: Not predicted by theory
  - v: Vacuum expectation value (VEV) of the Higgs field
- W boson masses: $m_{W^\pm} = \frac{1}{2} \cdot v \cdot g$
  - v: VEV
  - g: Coupling constant of $SU(2)_L \to$ weak coupling constant
- Z^0 boson mass: $m_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}$
  - g': Weak hypercharge coupling constant $g' = g \tan \theta_W$
- Photon mass: $m_\gamma = 0$ by construction

VEV not know (not predicted), so m_W, m_Z are not predicted. But Higgs mechanism predicts:

$$\frac{m_W}{m_Z} = \cos \theta_W$$

$\frac{m_W}{m_Z}$ can be derived by measurement. θ_W from measurements of branching ratios of leptonic Z^0 decays since $\frac{c_V}{c_A} c_A = 1 - 4 \sin \theta_W$ for the charged leptons.

$$c_V = I_W^{(3)} - 2Q \left(\sin \theta_W\right)^2$$
$$c_A = I_W^{(3)}$$

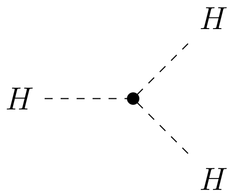
Measure VEV through Fermi's coupling constant

$$m_W = \frac{1}{2} v g \Rightarrow v^2 = \frac{4 m_W^2}{g^2} = \frac{1}{\sqrt{2} G_F}$$

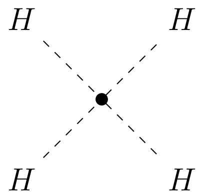
G_F: Measure from muon decays: $G_F = 1.166 \times 10^{-5} / \mathrm{GeV}^2 \Rightarrow V = 246\,\mathrm{GeV}$, the VEV of the Higgs field.

8.5 Coupling between the Higgs and other SM particles

From complete $SU(2) \times U(1)_Y$ Lagrangian: Higgs self-coupling:

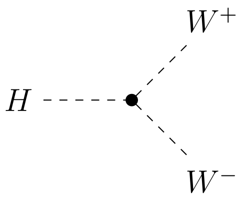


$$\lambda v \sim m_H^2$$



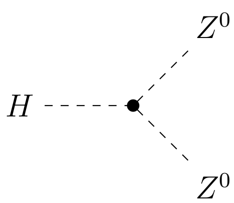
$$\frac{\lambda}{4} = \frac{m_H^2}{8v^2} = \frac{m_H^2}{32m_W^2}g^2 \sim m_H^2$$

$\Rightarrow$ Coupling strenth $\sim m_H^2$. Higgs boson to vector boson couplings:



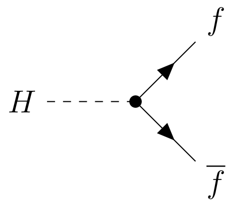
$$\frac{1}{2}g^2 v = gm_W$$

$\Rightarrow$ Proportional to m_W.



$$\frac{1}{4}\frac{g^2}{\cos\theta_W}v = \frac{g}{2}\frac{m_W}{(\cos\theta_W)^2}$$

$\Rightarrow$ Couling strength (for a fixed v) is proportional to the mass of the particle. Similarly for the photons:

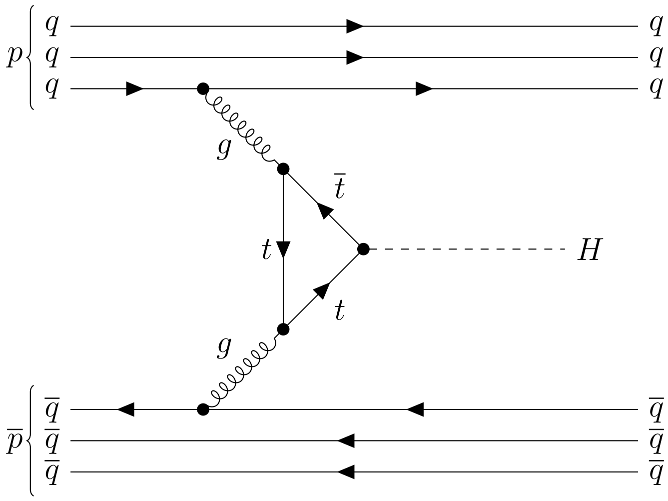


$\Rightarrow$ Copling strength proportional to m_f.

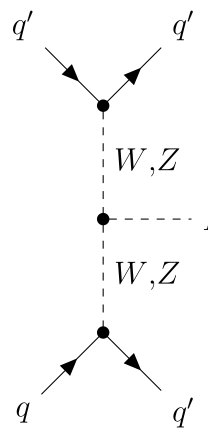
All the couplings can be tested by looking at Higgs decays.

8.6 Higgs boson production

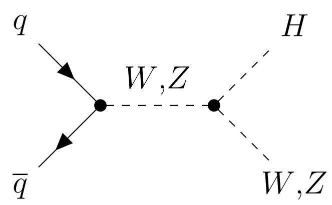
Production channels at the LHC:



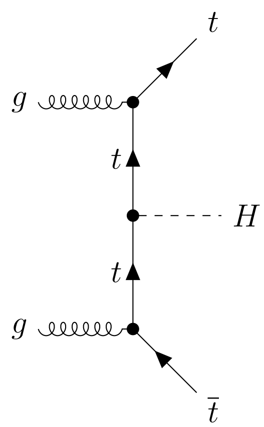
the gluon-gluon-fusion. The dominant quark in the loop is th t-quark since coupling of Higgs is $\sim m_f$.



the vector boson fusion.



the "Higgs Strahlung".



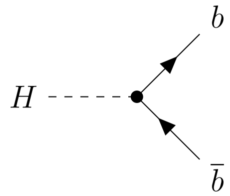
the associated Higgs production with top quark. At LHC energies ($\sqrt{s} = 14\,\text{TeV}$) the gluon-gluon fusion is the most probable process. Cross-section:

$$\sigma\,(pp \to Hx) \simeq 50\,\text{pb} \simeq 10 \times 10^{-10} \sigma(pp \to \text{anything})$$

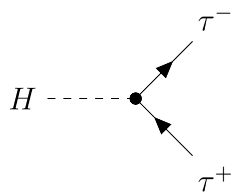
- Very tiny cross-section
- Have to deal with huge background and huge datasets

8.7 Higgs decays

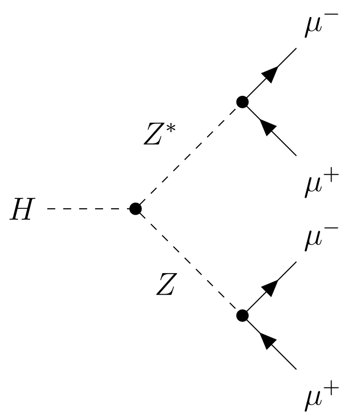
In principle the Higgs can decay to any SM particle (apart from t). But coupling proportional to mass. The dominant process for $m_H = 125\,\text{GeV}$.



$H \to b\bar{b}$: $BR = 58\,\%$ (can calculate from Feynman rules for $H \to b\bar{b}$ vertex). No decay into t since m_h is too small.

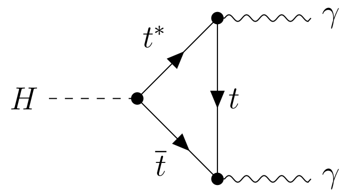


$H \to \tau^-\tau^+$: $BR = 63\,\% \simeq BR(H \to b\bar{b})\frac{m_\tau^2}{m_b^2}\frac{1}{3}$, $\frac{1}{3}$ because of the color charge. Detection of τ: $\tau^- \to \mu^- + \nu_\tau + \overline{\nu_\mu}$ and the muon is detected then in a muon detector.



$BR \simeq 2.6\,\%$.

Remark. One Z is off shell (virtual since $m_H < 2m_Z$). Clean signature (4μ). Very good mass reconstruction since all muon momenta are measured (no jets, no neutrinos,).



$H \to \gamma\gamma$: $BR \simeq 2 \times 10^{-3}$. Very clean signature, good mass reconstruction. Measurements of ATLAS, CMS:

$$m_H = 125.6(3)\,\text{GeV}$$

8.8 The standard model: Open questions and problems

25 free parameters:

- 9 fermion masses (+3 neutrino masses)
- 3 coupling constants (g, g', g_{QCD}) (α, G_f, α_s)
- 3 CKM mixing angles + 1 phase
- 3 PMNS mixing angles + 1 phase
- Higgs mass
- Higgs VEV $\to$ fixes m_W, m_Z

Problems:

- No explanation why electric charge is quantized (and why $Q(p) \equiv Q(e^-)$)
- No description for gravity
- no (good) Dark Matter candidate
- No explanation why the universe is matter dominated

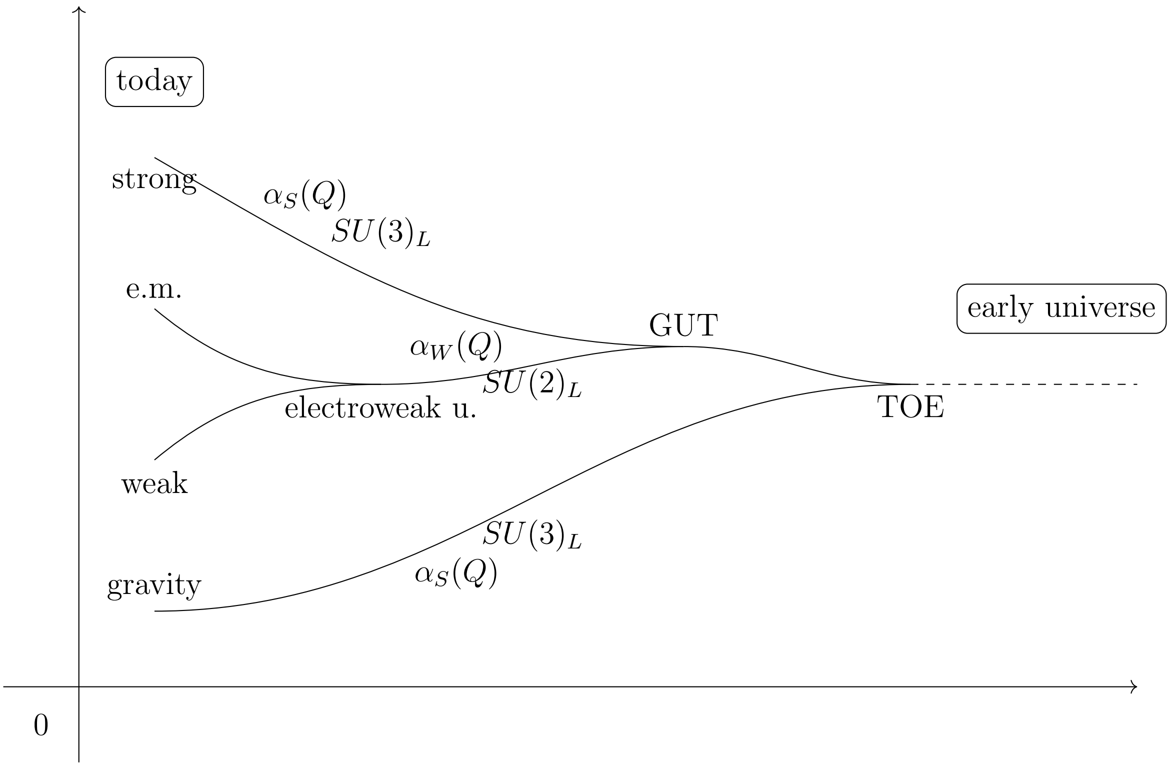
$\Rightarrow$ Is there a theory beyond the standard model? Ansatz: Proceed with unification of forces. We had the electroweak unification $SU(2)_L \times U(1)_Y$ corresponding to fundamental couplings: $g(SU(2)_L), g'(U(1)_Y)$ and $\frac{g'}{g} = \tan\theta_W$. Possible solution: Find symmetry group G

$$G \supset SU(2)_L \times U(1)_Y$$

which can predict the value of θ_W i.e. the relation between g and g'. True unification goes even further: Incorporate $SU(3)_L$ of QCD:

$$G \supset SU(3)_L \times SU(2)_L \times U(1)_Y$$

called GUTgrand unified theories. Consequence: Coupling constants g', g and g_{QCD} are effective coupling constants at low energy of fundamental coupling constant of G. At which energy can you expect such a single fundamental coupling? Look at running of couplings with energy:



Extrapolate the (known) running of the couplings:

$$m_{\text{GUT}} \simeq 10^{15}\,\text{GeV}$$

Still below the Planck Scale: $m_p = \sqrt{\frac{\hbar c}{G}} = 10^{19}\,\text{GeV}$. At these high energies gravity is also expected to be important as well. $\Rightarrow$ TOE: Theory of everything. Well-studied step towards a GUT: Supersymmetry. Proposal: There is a boson $\leftrightarrow$ fermion symmetry. Every SM particle has a superpartner with the same quantum numbers but spin.

Example 8.5. quark q (spin $\frac{1}{2}$) $\leftrightarrow$ squark $\tilde{q}$ (spin 0)

Example 8.6. electron e (spin $\frac{1}{2}$) $\leftrightarrow$ selectron $\tilde{e}$ (spin 0)

Example 8.7. gluon g (spin 1) $\leftrightarrow$ gluino $\tilde{g}$ (spin $\frac{1}{2}$)

Example 8.8. Higgs H (spin 0) $\leftrightarrow$ Higgsino $\tilde{H}$ (spin $\frac{1}{2}$)

No superpartners detected up to now.

$$\rightarrow m(\text{superpartner}) \geq 1\,\text{TeV}$$

We know $m(\text{SM particle}) << 1\,\text{TeV}$. $\rightarrow$ SUSY must be broken at current energy scale 1 TeV or higher. SUSY is interesting because:

- It solves the hierarchy problem “Why are SM particles s light compared to GUT scale?”: SUSY provides that through cancellations of self-corrections
- Provides massive new particles which could be the dark matter

→ All these things are very much beyond the scope of this lecture and will keep scientists busy for at least some decades.

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